Talk proposal

Mapping Temporal Correlations to Contextuality Correlations

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Overview

We are interested in a setup in which a sequence of measurements is performed on a single system, thereby producing a sequence of corresponding measurement outcomes. Measurements that can be performed on this system are collected into a set X, and the output of measurement m is taken to lie in the set O_m. Writing X^* for the set of all finite sequences of elements of X, we collect the sequences that can be performed on the system into a down-closed subset $\Sigma \subseteq X^*$ (with respect to the prefix order on X^*). We group this data together into a tuple $\mathcal{M} = \langle X, \Sigma, O \rangle$, which is called a **temporal measurement scenario**. A pair $\langle m, o \rangle$ with $m \in X$ and $o \in O_m$ is called a measurement event, and we write $\Delta_{\mathcal{M}}$ for the set of all measurement events associated with $\mathcal{M}$.

In contextuality scenarios, which are a spatial analogue of the current setup, there is a clear way in which to define *deterministic classicality* of a system: it corresponds to a measurement $m \in X$ being assigned the same outcome regardless of which other measurements are performed simultaneously; see e.g. the sheaf-theoretic treatment of contextuality [1]. For probabilistic systems, classicality in a sheaf approach sense is synonymous with the existence of a probabilistic local hidden variable model. That the assumptions of classicality hold experimentally can be guaranteed for Bell scenarios by appealing to special relativity, which has resulted in 'loophole-free' tests of Bell inequalities.

In the temporal case, assumptions of classicality are more subtle. The outcome of a measurement $m \in X$ may in principle depend on any of the previously performed measurements and their outcomes. So, a deterministic behaviour of the temporal system is instead described by a function

$$s : \Sigma \to O :: \sigma \mapsto o$$

which sends each sequence $\sigma \in \Sigma$ to an outcome $s(\sigma)$. The interpretation is that the last measurement of σ obtains the outcome $s(\sigma)$ when performed in the sequence σ, allowing for outcomes to depend on the whole history.[1] We refer to such functions as **global strategies**, and write $\mathcal{S}(\Sigma)$ for the set of all global strategies over Σ.

In practice, unbounded signalling of information from the past is not realistic and is limited by the **memory** of the system. For finite state machines—a paradigm of sequential computation—this limitation comes from the size of the state set. It therefore makes sense in this temporal setting to speak of classicality with respect to some memory bound. We consider a model of memory in which the system can only store a subset of the past measurements and their outcomes. We focus on three special cases, parametrised by for a fixed $k \in \mathbb{N}$:

1. the system remembers the k immediately preceding measurements;
2. the system remembers up to k of the previous measurements;
3. the system remembers the k immediately preceding measurement events (i.e. measurements and outcomes).

For example, given a sequence $\sigma = m_0 \dots m_n$, define $\mathsf{lookback}_k(\sigma) := m_{\max\{n-k+1,0\}} ... m_n$ to be its suffix of length $k+1$ (if it exists). Item 1 requires that any strategy $s \in \mathcal{S}(\Sigma)$ satisfy

$$\mathsf{lookback}_k(\sigma) = \mathsf{lookback}_k(\sigma') \implies s(\sigma) = s(\sigma').$$

We say that a strategy is $\mathsf{lookback}_k$-consistent if it satisfies this property. We write $\mathcal{S}_{\mathsf{lookback}_k}(\Sigma)$ for the set of $\mathsf{lookback}_k$-consistent strategies on Σ, noting that for any k it holds that

$$\mathcal{S}_{\mathsf{lookback}_k}(\Sigma) \subseteq \mathcal{S}_{\mathsf{lookback}_{k+1}}(\Sigma).$$

[1]Here, outcomes only explicitly depend on prior measurement choices and not on their outcomes. However, adding dependence on previous outcomes for s would be pointless: if a system is responding deterministically, knowing the full history of measurements is enough to reconstruct all past outcomes, and so they need not be included explicitly.

We similarly define functions $\Theta_k\colon \Sigma \to X^*$ and $\mathsf{lookback}_k^{(e)}\colon \Sigma \times \mathcal{S}(\Sigma) \to \Delta^*_{\mathcal{M}}$ such that a strategy acting according to item 2 is Θ_k-consistent, and according to 3 is $\mathsf{lookback}_k^{(e)}$-consistent (the superscript indicating that it is the past k measurements *events*). We denote the associated subsets of strategies $\mathcal{S}_{\Theta_k}(\Sigma)$ and $\mathcal{S}_{\mathsf{lookback}_k^{(e)}}(\Sigma)$.

Now, given a measurement scenario $\mathcal{M}$, performing a particular sequence of measurements $\sigma \in \Sigma$ produces a probability distribution e_σ on corresponding output sequences, i.e. on assignments of outcomes to the set $\downarrow \sigma$ of prefix sequences of σ. The collection $e = (e_\sigma)_{\sigma\in\Sigma}$ is called an **empirical model**.

An empirical model which for each σ has support only on the $\mathsf{lookback}_k$-consistent strategies is said to be $\mathsf{lookback}_k$-consistent, and we write $\mathsf{EM}^{(\mathsf{lookback}_k)}(\mathcal{M})$ for the convex set of such models. We can make a similar definition based on Θ_k and $\mathsf{lookback}_k^{(e)}$. Note that Θ_k-consistent empirical models are restricted to use the available memory according to the fixed choice of function Θ_k. More generally we would like to describe systems which may respond with any strategy that stores at most k of the past measurements. To this end, we say that a model is L_k-consistent if every strategy in its support is Θ_k-consistent for some Θ.

So, in general, we write $\mathsf{EM}^{(F)}(\mathcal{M})$ for the set of F-consistent empirical models with $F \in \{\mathsf{lookback}_k, L_k, \mathsf{lookback}_k^{(e)}\}$. The function F introduces additional no-signalling constraints for the empirical model e, which go beyond the usual arrow of time constraints and are not captured by restrictions of the presheaf $\mathcal{D}_R \circ \mathcal{S}_F$. Such constraints reflect the fact that signalling from the past has been in some way restricted.

While in the spatial case classicality is synonymous with the existence of a local hidden variable theory, here we take classicality of an F-consistent empirical model to mean realisability by a classical machine E_F that produces F-consistent strategies according to some probability distribution on global strategies $h \in \mathcal{D}_R \circ \mathcal{S}_F(\Sigma)$. We say that e is **E_F-classical.**[2]Thus for e E_F-nonclassical, *local* probability distributions are consistent with those that E_F produces, but there is no extension of these to a consistent *global* distribution (as with contextuality). Note that defining classicality with respect to memory bounds on a classical machine, as captured by a restriction on strategies F, avoids hypothesising about what it means for temporal correlations to be classical (as in Leggett and Garg's macrorealistic assumptions, which suppose that noninvasiveness should hold). This circumvents a debatable philosophical issue by appealing to resource-theoretic notions, where in this case the resource of interest is memory.

An advantage of casting the setup in a sheaf-theoretic framework is that when the system stores only measurements in memory, i.e. in the cases $F = \mathsf{lookback}_k$ or $F = L_k$, we are able to map temporal measurement scenarios to a particular type of contextuality setup, in which correlations are no longer temporal but spatial. Nonclassical temporal empirical models are then the pullback of contextual empirical models on this image scenario. We first show that strategies $f \in \mathcal{S}_F(U)$ are in one-to-one correspondence with sections $s \in \mathcal{E}(\mathcal{C}_F(U))$, where

$$\mathcal{C}_F(U) := \{F(\sigma) | \sigma \in U\}.$$

Calling this bijection α, then given an empirical model $\{w_C\}_{C\in\Sigma_{\mathcal{C}_F(\mathcal{M})}}$ we define a temporal empirical model

$$e_C(f) := w_{\mathcal{C}_F(C)}(\alpha(f)).$$

This forms the first theorem of the paper, stated below.

Theorem 1. *For $F \in \{\mathsf{lookback}_k, L_k\}$ there is a map $\mathcal{C}_F$ from temporal measurement scenarios to contextuality measurement scenarios, such that empirical models $w \in \mathsf{EM}(\mathcal{C}_F(\mathcal{M}))$ can be pulled back via $\mathcal{C}_F^*$ to F-consistent empirical models $\mathcal{C}_F^* w \in \mathsf{EM}^{(F)}(\mathcal{M})$ on the temporal measurement scenario $\mathcal{M}$. This map preserves and reflects nonclasicality, meaning that an empirical model w on $\mathcal{C}_F(\mathcal{M})$ is contextual if and only if $\mathcal{C}_F^* w$ is E_F-nonclassical.*

Vorob'ev's theorem and quantum advantage

The advantage of mapping an empirical model of the temporal type to one of the contextual type is that the latter have been extensively studied in the sheaf-theoretic approach. Therefore, there are a number of well-developed tools we can readily utilise on contextuality scenarios. Crucially, as the map both preserves and reflects contextuality, it can be used to transfer results, allowing us to say something in turn about the classicality of empirical models in the original temporal scenario.

With this in mind, we recall an application of Vorob'ev's Theorem, originating in game theory [3], to contextuality measurement scenarios, which was studied in [4]. This theorem characterises the contextuality measurement scenarios that admit contextual empirical models.

[2]An E_F machine is for example a finite state machine which uses its state set to store past measurements and outcomes. Note that contextuality and finite state machines has been studied too in [2].

Theorem 2 ([4, 3])**.** *Let $\mathcal{M} = \langle X_{\mathcal{M}}, \Sigma_{\mathcal{M}}, O_{\mathcal{M}} \rangle$ be a contextuality measurement scenario where Σ is a simplicial complex representing the compatibility relation of elements in X. Then all empirical models defined on $\mathcal{M}$ are non-contextual if and only if $\Sigma_{\mathcal{M}}$ is acyclic.*

Acyclicity here means that one can remove the measurements from the scenario one by one in such a way that the removed measurement at each stage belongs to a single maximal context, i.e. all the measurements compatible with it are jointly compatible.

The following corollary follows straightforwardly from Theorems 1 and 2.

Corollary 3. *Let $\mathcal{M}$ be a temporal measurement scenario. Every F-consistent empirical model on $\mathcal{M}$ is E_F-classical if and only if the corresponding contextual measurement scenario $\mathcal{C}_F(\mathcal{M})$ has acyclic simplicial complex $\Sigma_{\mathcal{C}_F(\mathcal{M})}$.*

Note that we obtain both directions of Vorob'ev's Theorem, so that a non-acyclic $\Sigma_{\mathcal{C}_F(\mathcal{M})}$ implies that there exists an E_F-nonclassical empirical model $e \in \mathsf{EM}^{(F)}(\mathcal{M})$.

Although the constructed map breaks down when memory storage of outputs is allowed, we are nevertheless able to show that $\Sigma_{\mathcal{C}_{\mathsf{lookback}_k}(\mathcal{M})}$ not acyclic also implies the existence of an empirical model that is $\mathsf{lookback}_k^{(e)}$-consistent but $\mathsf{E}_{\mathsf{lookback}_k^{(e)}}$-nonclassical. This is done by constructing an empirical model which is deterministic at certain measurements, so that allowing for strategies which store outputs of these measurements is not more advantageous. This extends one direction of Corollary 3 to $\mathsf{lookback}_k^{(e)}-$consistent strategies.

Theorem 4. *Let $\mathcal{M}$ be a temporal measurement scenario. If the measurement scenario $\mathcal{C}_{\mathsf{lookback}_k}(\mathcal{M})$ has a non-acyclic simplicial complex, then there exists a $\mathsf{lookback}_k^{(e)}$-consistent empirical model on $\mathcal{M}$ which cannot be generated by a classical machine $\mathsf{E}_{\mathsf{lookback}_k^{(e)}}$.*

Conclusion and Future Work

The assumption underlying the F-consistent models we consider is that the system at any given time can only remember some subset of the past measurements that have been performed and outputs obtained (as specified by F). In our paper we show that

1. When only measurements are stored, we can map temporal scenarios to contextuality scenarios via $\mathcal{C}_F$ in way that preserves and reflects nonclassicality. This allows the identification of empirical models which have *local* support on deterministic strategies that a classical machine E_F can produce, but which are inconsistent with any *global* distribution on such strategies. We call this behaviour E_F-nonclassial.
2. This map does not work when outputs are stored. Nevertheless,
   (a) An empirical model generated by a classical machine $\mathsf{E}'_{F'}$ that can store past measurements and their outcomes can be realised by a machine E_F that stores only measurements if one increases the allowed number of stored measurements. We analyse this trade-off for certain examples of empirical models.
   (b) We can for any measurement scenario which admits $\mathsf{E}_{\mathsf{lookback}_k}$-nonclassical empirical models find at least one empirical model which is also $\mathsf{E}'_{\mathsf{lookback}_k^{(e)}}-$nonclassical.

In future work we hope to extend these notions to causal scenarios studied in [5], including the causal Bell scenarios studied in [6].

References

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