From dual-unitary to biunitary: a 2-categorical model for exactly-solvable many-body quantum dynamics

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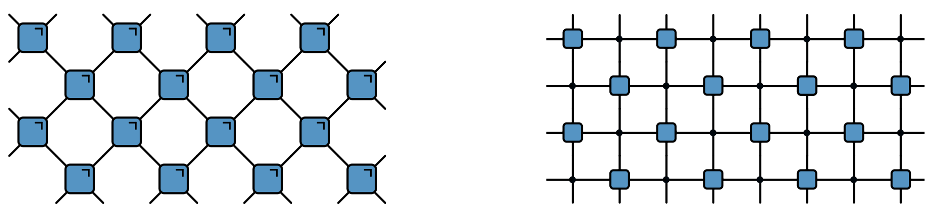
Summary of `https://arxiv.org/abs/2302.07280`.

The dynamics of isolated many-body quantum systems remains a complex problem, and exact solutions are both scarce and typically non-representative of the chaotic dynamics of generic quantum systems. In recent years unitary circuits have have been studied intensively as models of unitary many-body dynamics governed by local interactions [1–8]. Such circuits mimic the dynamics generated by a local Hamiltonian on a one-dimensional lattice, with 'dynamics' taking place in discrete time.

A special class of *dual-unitary* circuits was recently identified, characterized by the property that the dynamics remains unitary when exchanging the roles of space and time [9, 10]. This duality endows the circuits with many remarkable properties, including soluble correlation dynamics [9, 11–14], out-of-time-order correlators [15, 16], maximal entanglement growth [11, 17–19], and analytically tractable signatures of quantum chaos [20–28].

Most recent studies of dual-unitary dynamics have focused on 2-site dual-unitary gates arranged in a regular brickwork pattern (below, left). An alternative model has recently been proposed by Prosen [29] who showed that dual-unitary interactions could also be arranged 'round-a-face' — a model we name *clockwork* — with a circuit representation involving 1-site unitaries controlled by both neigbours (right):



(1)

(a) Conventional brickwork circuit *(b) Conventional clockwork circuit*

These models have been shown to have similar global properties, supporting notions of unitarity along both the time (vertical) and space (horizontal) direction, and with vanishing single-site correlation functions everywhere except on the causal light cone. Prosen has raised the question of the nature of the relationship between brickwork and clockwork circuits, to which our paper provides a definitive answer.

Here we present such a generalisation based on *biunitary connections*, algebraic structures with a variety of applications in 2-categorical linear algebra. We reason about these structures using the *shaded calculus* [30–32], a powerful graphical system analogous to traditional tensor notation, but with the added feature of shadings assigned to certain diagram regions. In the simplest representation scheme, these shadings represent the assignment of a *finite indexing set* to the region, and any wires and vertices that border the region are then indexed by that set. If any particular region is left unshaded, that means the region is assigned a 1-element set; in this way the shaded calculus subsumes the traditional tensor notation, since the wires and vertices bordering such a region become trivially indexed. For every shaded

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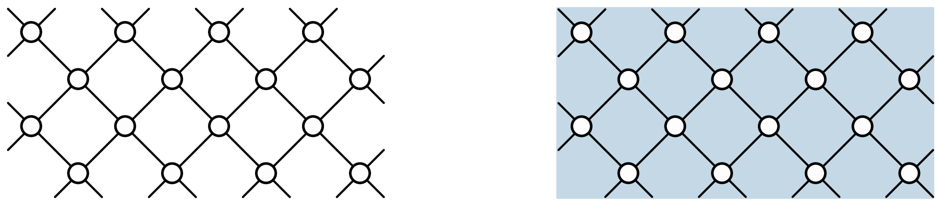
calculus diagram we can compute a traditional representation in tensor notation. More importantly, any such diagram has an explicit representation in terms of (controlled) gates.

Full details of the shaded calculus are given in the preprint, but we can readily give a clear intuitive sense of our main results. Imposing for now spacetime homogeneity, there are two possible shaded circuit structures, either completely unshaded or completely shaded, as follows:



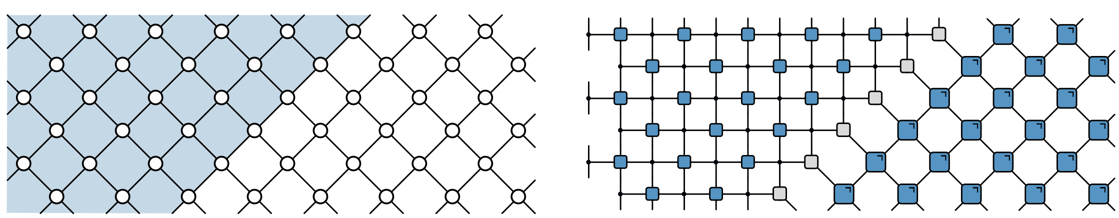
(2)

(a) Shaded calculus brickwork circuit *(b) Shaded calculus clockwork circuit*

Both these diagrams are drawn in the shaded calculus, which we indicate by using circles for the vertices, to contrast with the conventional circuit diagrams where we draw vertices as squares. Diagram (2)(a) is a degenerate case of the shaded calculus since every region is trivially shaded; it can therefore be interpreted directly as an ordinary tensor diagram, yielding the traditional brickwork circuit (1)(a). In contrast, diagram (2)(b) has nontrivial shading, and computing its associated tensor representation gives precisely the clockwork circuit (1)(b).

This shaded calculus representation therefore yields a structural unification of the brickwork and clockwork models; while as conventional circuits they have very different structures, in the shaded calculus their representation is uniform, with 4-valent vertices stacked in a brickwork pattern. Furthermore, the dual unitarity property, which requires different definitions in the conventional brickwork and clockwork cases, is unified by the single concept of biunitarity in the shaded calculus representation.

However, our scheme offers more than notational unification, since spacetime homogeneity is not a requirement. Dropping this condition allows us to explore a rich family of exactly-solvable circuits, which can include more complex interaction patterns and geometries, which have not previously been described. In the following we give the shaded calculus representation on the left, and the conventional circuit on the right. As a first example, we consider a simple diagonal boundary between clockwork and brickwork regions:



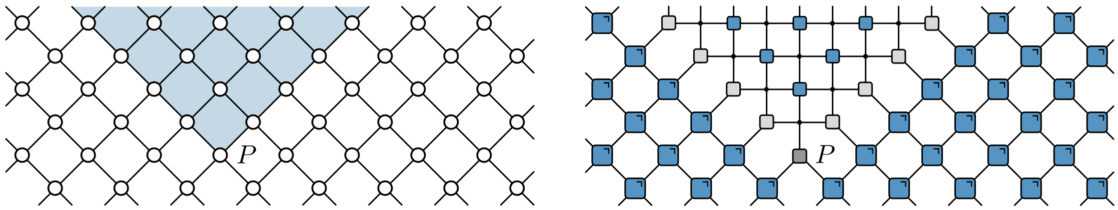
(3)

Shaded calculus representation *Unitary circuit representation*

This describes a *dynamical boundary* between a clockwork region on the left of each picture, and a brickwork region on the right, separated by a boundary which moves left-to-right over time. In the diagram on the left, along the diagonal boundary, we see a new sort of biunitary vertex, with two adjacent shaded regions in its neighbourhood, and two adjacent unshaded regions. Previous results on biunitary connections tell us that such a vertex encodes the data of a *quantum Latin square*[1] [33], combinatorial objects of recent interest in quantum foundations.

[1]A quantum Latin square is an n-by-n array of vectors $\in \mathbb{C}^n$, where every row and column is an orthonormal basis.

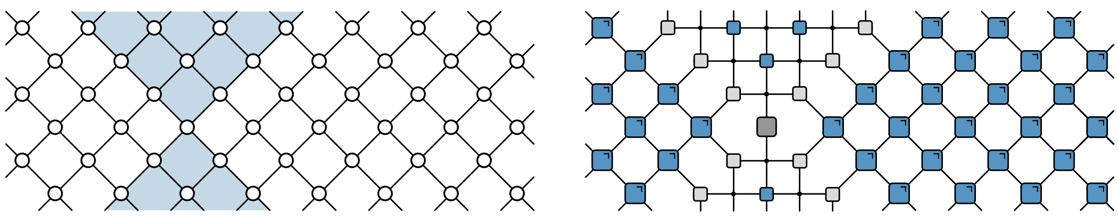
Other interesting phenomena are possible. In our second example we consider an inverted triangular clockwork region with its apex at a certain spacetime point *P*:



(4)

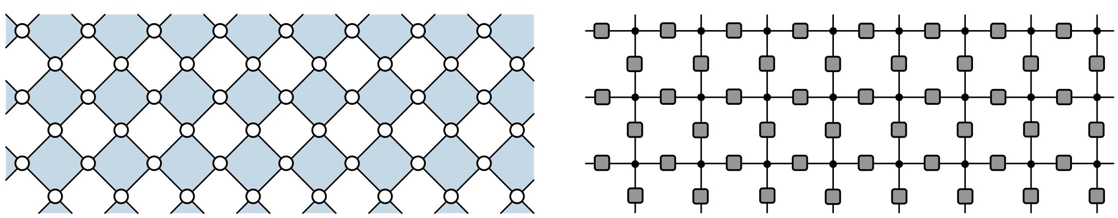
At point *P* we see another shading pattern around the biunitary vertex, with one shaded and three unshaded regions. These biunitaries have been characterized as corresponding to *unitary error bases*[2] [34, 35], another important quantum combinatorial structure, introduced originally by Werner [36] for classifying quantum teleportation protocols.

A third example is the reflection of two incident boundaries with opposite velocities:


(5)

Here the biunitarity property implies that the central reflection point is described by a complex Hadamard matrix, a result originally demonstrated by Jones in his work on subfactor theory [32].

A final example suggested by our approach is the construction of homogenous circuits with a regular shading pattern, such as the following example using the Hadamard biunitary:


(6)

Circuits of this form already appeared in the literature as *ad hoc* decompositions of unitary circuits representing the 'self-dual kicked Ising model' [10, 25, 37–39].

Crucially, we show that all circuits produced by these techniques satisfy a range of attractive properties already known piecemeal for the conventional brickwork and/or clockwork models. As a novel application we use a definition of solvable initial states appropriate for the shaded calculus to exactly describe entanglement dynamics for clockwork circuits.

[2]A unitary error basis is family of unitary matrices which form an orthogonal basis of the operator space.

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