Coarse-graining and compounding as monads
Extended Abstract for QPL 2024
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1 Introduction

Two very basic constructions involving general probabilistic models are the formation of coarse-grained versions of experiments, and the formation of branching sequential experiments. The latter allows for the conditioning of states on the results of previous measurements. When one condition on the results of different coarse-grainings of the same previous experiment, the possibility of interference effects arises. Both constructions can be understood as monads on a suitable category **Prob** of (general) probabilistic models. Moreover, these are connected by a distributive law, allowing for a composite monad describing the closure of a probabilistic model under both coarse-graining and sequential measurement. Algebras for all three monads are discussed. Under weak additional assumptions, algebras for the coarse-graining monad are always coherent and algebraic, and thus give rise to complete orthomodular posets.

Algebras for the combined monad appear particularly interesting. For one thing, they allow for interference effects, in a sense that includes quantum interference as a special case. Subject to the weak conditions mentioned aove, these models give rise to complete orthomodular posets carring an associative binary operation distributing over orthogonal joins.

What follows is a detailed sketch. A more complete account will appear elsewhere.

2 Probabilistic Models

Different authors mean slightly different things by a "general(ized) probabilistic theory". The framework sketched here draws on work of Foulis and Randall in the 1970s and 80s (see, e.g., [6, 7]), enhanced with some category-theoretic ideas. General references for this section are the survey papers [3, 15].

A **test space** is a nonempty collection $\mathcal{M}$ of nonempty sets $E, F, ...$, each regarded as the outcome set of some operation, experiment, or *test*. The set $X := \bigcup \mathcal{M}$ is the *outcome-space* of $\mathcal{M}$. A *probability weight* on $\mathcal{M}$ is a function $\alpha : X \to [0, 1]$ summing to unity on each test. A *probabilistic model* — or simply a *model*, for short — is a pair $A = (\mathcal{M}, \Omega)$ where $\mathcal{M}$ is a test space and Ω is a distinguished set of probability weights on $\mathcal{M}$. I write $\mathcal{M} = \mathcal{M}(A)$, $X = X(A)$, and $\Omega = \Omega(A)$ to indicate the test space, outcome-space, and and *state space* of a model A. Note that if $\mathcal{M}$ is any test space, we have an associated model $(\mathcal{M}, \Pr(\mathcal{M}))$ where $\Pr(\mathcal{M})$ is the set of all probability weights on $\mathcal{M}$.

Standing Assumption: To avoid trivialities, I will make it a standing assumption that the set of states Ω is *positive*, meaning that for every $x \in X$, there is at least one $\alpha \in \Omega$ with $\alpha(x) > 0$. It follows that $\mathcal{M}$ is *irredundant*, that is, an antichain: if $E, F \in \mathcal{M}$ and $E \subseteq F$, then $E = F$. (Note that this also constrains the set of test spaces I consider, since there are simple examples of test spaces $\mathcal{M}$ with $\Pr(\mathcal{M}) = \emptyset$.)

The following examples indicate how this framework accommodates both classical and quantum probability theory.

Example 1: classical models Let (S, Σ) be a measurable space. The set $\mathcal{M}(S, \Sigma)$ of countable measurable partitions of S is a test space. The probability weights on this correspond in an obvious way to probability measures on (S, Σ). A *classical* probabilistic model is one of the form $(\mathcal{M}(S, \Sigma), \Omega)$ where Ω is a set of probability measures on (S, Σ).

Example 2: Hilbert models If $\mathbf{H}$ is a Hilbert space $\mathbf{H}$, let $X(\mathbf{H})$ denote the unit sphere of $\mathbf{H}$ and $\mathcal{F}(\mathbf{H})$, the collection of frames (unordered orthonormal bases) for $\mathbf{H}$, regarded as a test space. Every density operator W on $\mathbf{H}$ gives rise to a probability weight on $\mathcal{F}(\mathbf{H})$ defined by $\alpha_W(x) = \langle Wx, x \rangle$ (Gleason's Theorem asserts that if $\dim(\mathbf{H}) > 2$, then every probability weight has this form.) By a *Hilbert model*, I mean a model $(\mathcal{F}(\mathbf{H}), \Omega)$ where Ω is a set of (states associated with) density operators.

Example 3: von Neumann models Let $\mathcal{A}$ be a von Neumann algebra without and let $\mathbb{P}(\mathcal{A})$ be its projection lattice. The set $\mathcal{M}(\mathcal{A})$ of countable (resp., finite) partitions of unity in $\mathbb{P}(\mathcal{A})$ is a test space, and every normal (resp., arbitrary) state $f \in \mathcal{A}^*$ induces a probability weight on it. (If $\mathcal{A}$ has no type I_2 factor, then conversely, every By a *von Neumann model*, I mean one of the form $(\mathcal{M}(\mathcal{A}), \Omega(\mathcal{A}))$ where $\mathcal{A}$ is a von Neumann algebra and $\Omega(\mathcal{A})$ is a set of normal states on $\mathcal{A}$.

Events and Perspectivity An *event* for a test space $\mathcal{M}$ is simply an event in the usual probabilistic sense for one of the tests in $\mathcal{M}$; that is, an event is a set $a \subseteq E$ for some $E \in \mathcal{M}$. We write $\mathcal{E}(\mathcal{M})$ for the set of all events fo $\mathcal{M}$. If α is a probability weight on $\mathcal{M}$, we define the probability of an event a in the usual way, that is, $\alpha(a) = \sum_{x \in a} \alpha(x)$. In particular, $\alpha(E) = 1$ for every test $E \in \mathcal{M}(A)$. Conversely, if $E \in \mathcal{E}(A)$ with $\alpha(E) = 1$, then E is a test. (Indeed, if $E \subsetneq F \in \mathcal{M}$, then by our positivity assumption, there exists some $y \in F \setminus E$, and thus, some state $\alpha \in \Omega(A)$ with $\alpha(y) > 0$, whence, $\alpha(F) > 1$, a contradiction.)

Two events $a, b \in \mathcal{E}(\mathcal{M})$ are *orthogonal*, written $a \perp b$, iff they are disjoint and their union is still an event. If $a \perp b$ and $a \cup b \in \mathcal{M}$ — that is, if a and b partition a test — then a and b are *complementary*, and we write a co b. If two events a, b are both complementary to an event c, we say that a and b are *perspective*, and write $a \sim b$.

Algebraic test spaces and orthoalgebras It is easy to check that if $a \sim b$, then $\alpha(a) = \alpha(b)$ for all probability weights α on $\mathcal{M}$. Also, owing to irredundance, if $a \subseteq b \sim a$, we have $a = b$. $\mathcal{M}$ is *algebraic* [15] iff, for all events $a, b, c \in \mathcal{E}(\mathcal{M})$, if $a \sim b$ and b is complementary to c, then a is also complementary to c. If $\mathcal{M}$ is algebraic, $\sim$ is an equivalence relation on $\mathcal{E}(\mathcal{M})$, with the feature that

$$a \sim b \perp c \ \Rightarrow a \perp c \text{ and } a \cup c \sim b \cup c.$$

This makes it possible to define a partial binary operation $\oplus$ on the set $\Pi = \Pi(\mathcal{M})$ of equivalence classes of events by setting

$$[a] \oplus [b] = [a \cup b] \text{ when } a \perp b.$$

The structure $(\Pi, \oplus)$ is then an orthoalgebra [**?**] called the *logic* of $\mathcal{M}$. Every probability weight on $\mathcal{M}$ descends to a probability measure on Π, and every orthoalgebra arises as $\Pi(\mathcal{M})$ for some algebraic test space $\mathcal{M}$. (Indeed, if L is an orthoalgebra, let $\mathcal{D}(L)$ denote the set of orthopartitions of its unit: this is an algebraic test space, and $L \simeq \Pi(\mathcal{D}(L))$.) We'll say that a model A is algebraic iff $\mathcal{M}(A)$ is algebraic, in which case we write $\Pi(A)$ for $\Pi(\mathcal{M}(A))$.

Linearized Models and Effect Algebras If A is a probabilistic model, let $\mathbf{V}(A)$ denote the subspace of $\mathbb{R}^{X(A)}$ spanned by A's state-space, $\Omega(A)$. Ordered pointwise, $\mathbf{V}(A)$ is a conebase space: every element of $\mathbf{V}(A)_+$ is a non-negative multiple of a probability weight, and every element of $\mathbf{V}(A)$ is the difference of two elements of $\mathbf{V}(A)_+$. The dual, $\mathbf{V}(A)^*$, of $\mathbf{V}(A)$, with the dual ordering, is an order-unit space with unit effect u given by $u_A(\alpha) = \sum_{x \in E} \alpha(x)$ for any $E \in \mathcal{M}(A)$. A functional $a \in \mathbb{E}(A)$ with $0 \leq a \leq u_A$ is called an *effect*, and the collection $[0, u_A]$ of effects is a basic example of an *effect algebra*. Every event $a \in \mathcal{E}(A)$ gives rise to an effect $\widehat{a}$ by evaluation, i.e., $\widehat{a}(\alpha) = \alpha(a)$, but in general there will be effects not of this form.

3 Categories of probabilistic models

Probabilistic models can be made into a category in various ways, depending on what kinds of maps one takes as morphisms (see, e.g., [6, 15]). The following will be general enough for our purposes:

Definition: If A and B are probabilistic models, a morphism from A to B is a test-space morphism $\phi : \mathcal{M}(A) \to \mathcal{M}(B)$ such that For each state $\beta \in \Omega(B)$,

$\phi^*(\beta) := \beta \circ \phi$ belongs to $\mathbf{V}(A)$.

Definition: Let A and B be probabilistic models. A *morphism* $\phi : A \to B$ is a mapping $\phi : X(A) \to X(B)$ such that

(a) For each test $E \in \mathcal{M}(A)$, $\phi(E) \in \mathcal{E}(B)$;

(b) For all $x, y \in X(A)$, $x \perp y \Rightarrow \phi(x) \perp \phi(y)$;

(c) For each state $\beta \in \Omega(B)$, $\phi^*(\beta) := \beta \circ \phi$ belongs to $\mathbf{V}(A)$.

Remarks: Condition (a) implies that ϕ maps $\mathcal{E}(A)$ into $\mathcal{E}(B)$, giving us a functor $\mathcal{E} : \mathbf{Prob} \to \mathbf{Set}$. Condition (b) can be rephrased as saying that morphisms are *locally injective*, i.e., injective on every test.

Clearly, the composition of two morphisms, whether of test spaces or of models, is again morphism, and the identity mapping on outcomes provides an identity morphism. Thus, we have a category, **Prob**, of probabilistic models and their morphisms.

Conditions (a) and (b) together define a reasonable notion of a morphism of test spaces. Note that for models of the form $(\mathcal{M}, \Pr(\mathcal{M}))$, condition (c) is automatic. Thus, we also have a category **Test** of test space and test-space morphisms, and an adjunction between these, given by the functor U : **Prob** $\to$ **Test** taking A to $\mathcal{M}(A)$, and the functor Pr**Test** $\to$ **Prob** taking $\mathcal{M}$ to $(\mathcal{M}, \Pr(\mathcal{M}))$ (and both acting as the identity on morphisms). It's easy to check that Pr is left-adjoint to U.

We will be interested below in some special classes of morphisms. Specifically,

Definition: A morphism $A \to B$ is

(i) *strong* iff $\phi(E) \sim \phi(F)$ for all $E, F \in \mathcal{M}(A)$, and

(ii) *test-preserving* iff $\phi(E) \in \mathcal{M}(B)$ for every test $E \in \mathcal{M}(A)$, and

(iii) an *embedding* iff test-preserving and (gloabally) injective.

Lemma 1: *A morphism $\phi : A \to B$ is strong iff it preserves perspectivity, i.e.,*

$$a \sim b \;\Rightarrow\; \phi(a) \sim \phi(b)$$

for all events $a, b \in \mathcal{M}(A)$.

Proof: Clearly, perspectivity-preserving implies strong. For the converse, suppose σ is strong. If a co c and c co b, then $\phi(a) \perp \phi(c)$, $\phi(b) \perp \phi(c)$, and $\phi(a) \cup \phi(c) \sim \phi(c) \cup \phi(b)$. Hence, for some event $e \in \mathcal{E}(B)$, $\phi(a) \cup \phi(c)$ co e and $\phi(b)$ co $\phi(c)$ co e, whence, $\phi(a) \sim \phi(b)$ with axis $\phi(c) \cup e$.

It is straightforward that a test-preserving morphism is strong. The following shows that left inverses of embeddings are test-preserving.

Lemma 2: *Let $\phi : A \to B$ be an embedding, and let $\psi : B \to A$ be a morphism with $\psi \circ \phi = id_B$. Then ψ is test-preserving.*

Proof: If $\alpha \in \Omega(A)$ and $E \in \mathcal{M}(A)$, we have $\phi(E) \in \mathcal{M}(B)$, so

$$\psi^*(\alpha)(\phi(E)) = \alpha(E) = 1$$

It follows that $\psi^*(\alpha)$ is a probability weight in $\Omega(B)$. Thus, if $F \in \mathcal{M}(B)$, we have

$$\alpha(\psi(F)) = \psi^*(\alpha)(F) = 1,$$

so $\psi(F) \in \mathcal{M}(A)$. □

It is also straightforward that if $\phi : A \to B$ and $\psi : B \to A$ are mutually inverse morphisms, then $\phi : X(A) \to X(B)$ is a bijection, ψ is its inverse, and both ϕ and ψ are test-preserving. In other words, an *isomorphism* of models A and B is just what we'd expect: a bijection $X(A) \to X(B)$ preserving tests, and states, in both directions.

4 The Coarsening Monad

Operationally, it is always possible to "coarse-grain" a test E by partitioning it: when an outcome of E is obtained, one records only the corresponding cell of the partition.

Definition: The **coarsening** of a test space $\mathcal{M}$ is the test space $\mathcal{M}^{\#}$ consisting of all partitions of tests in $\mathcal{M}$.

The outcome-set of $\mathcal{M}^{\#}$ is $\mathcal{E}(\mathcal{M}) \setminus \{\emptyset\}$. Every probability weight on $\mathcal{M}$ lifts to a probability weight on $\mathcal{M}^{\#}$ given by $\alpha(a) = \sum_{x \in a} \alpha(x)$, and every probability weight on $\mathcal{M}^{\#}$ has this form for a probability on $\mathcal{M}$. The coarsening of a model A is the model $A^{\#}$ where
(a) $\mathcal{M}(A^{\#}) = \mathcal{M}(A)^{\#}$
(b) $\Omega(A^{\#})$ consists of all lifts of states in $\Omega(A)$ to probability weights on $\mathcal{M}^{\#}(A)$.

There is a canonical embedding $\phi : A \to A^{\#}$, namely $\phi : x \mapsto \{x\}$ for all $x \in X(A)$.

The following is straightforward:

Lemma 3: *If A is algebraic, so is $A^{\#}$, and in this case there is an isomorphism $\phi : \Pi(A) \simeq \Pi(A^{\#})$ given by $\phi([a]) = [\{a\}]$ where $[a] \in \Pi(A)$ is the perspectivity class of a in $\mathcal{E}(A)$, and $[\{a\}] \in \Pi(A^{\#})$ is the perspectivity class of $\{a\}$ in $\mathcal{E}(A^{\#})$.*

Lemma 4: *Let $\phi : A^{\#} \to A$ be a morphism such that $\phi(\{x\}) = x$ for every $x \in X(A)$. Then*
(a) ϕ is test-preserving

(b) For all $a \subseteq E \in \mathcal{M}(A)$, $(E \setminus a) \cup \{\phi(a)\} \in \mathcal{M}(A)$. In particular, $\{\phi(a)\} \sim a$.

Proof: (a) follows directly from Lemma (1). For part (b), let $E' = \{a\} \cup \{\{x\} | x \in E \setminus a\}$, and use the fact that ϕ is test-preserving. □

If $\phi : A \to B$ is a morphism, then $\phi^{\#}(a) := \phi(a) = \{\phi(x) | x \in a\}$ defines a morphism from $A^{\#}$ to $B^{\#}$. It's easy to check that this yields an endofunctor # on **Prob**. In fact, it is a monad. The unit and multiplication are given by the morphisms

$$\eta_A : A \to A^{\#}, \ \eta_A(x) = \{x\}$$

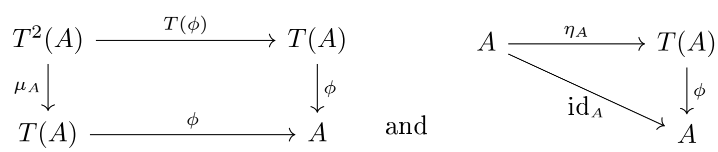
and

$$\mu_A : A^{\#\#} \to A^{\#}, \ \mu_A(a) = \bigcup a$$

for any $x \in X(A)$ and $a \in \mathcal{E}(A^{\#\#})$.

#-Algebras, Coherences, and Cohesions

Recall that an *algebra* for a monad T is an object A plus a morphism $\phi : T(A) \to A$ such that the diagrams



(I) (II)

commute. Unpacking this in the case where $T = \#$, we find that a morphism $\sigma : A^{\#} \to A$ turns A into a *#-algebra* iff

(i) $\sigma(\bigcup_i a_i) = \sigma\{\sigma(a_i)\}$ for jointly orthogonal sets $\{a_i\}$ of events;

(ii) $\sigma(\{x\}) = x$ for all $x \in X(A)$.

Call such a morphism a *coherence* on A. A #-algebra, then, is a model with a designated coherence.

Examples The classical model $A(S, \Sigma)$ associated with a measurable space has a coherence given by $\sigma(\{a_i\}) = \bigcup_i a_i$ where $\{a_i\}$ is any pairwise-orthogonal collection of nonempty sets $a_i \in \Sigma$. The von Neumann model $A(\mathcal{A})$ has a coherence given by $\sigma(\{p_i\}) = \bigoplus p_i$ where $\{p_i\}$ is any pairwise orthogonal family of projections in $\mathcal{A}$.

Remark: In quantum-mechanical usage, in this last case $\oplus_i p_i$ is a *coherent* combination of the projections $p \in a$. The distinction is crucial in sequential experiments, where interference effects only manifest themselves when one conditions on coherent combinations [17]

By Lemma 4, a coherence σ is always test-preserving, hence, a strong morphism, and satisfies $\sigma(a) \sim a$ for every $a \in \mathcal{E}(A)$. It follows that if $\mathcal{M}(A)$ is algebraic, σ also satisfies

$$\sigma(a) \in E \Rightarrow (E \setminus \{\sigma(a)\}) \cup a \in \mathcal{M}(A). \tag{1}$$

Definition: A coherence σ on A satisfying (1) is a *cohesion*. A #-algebra (A, σ) is *cohesive* iff σ is a cohesion.

Call a test space *projective* iff it satisfies the condition $x \sim y \Rightarrow x = y$ for all outcomes $x, y \in X(A)$. The test spaces $\mathcal{M}(S, \Sigma)$ and $\mathcal{M}(\mathcal{A})$ associated with a measureable space or a von Neumann algebra are both projective, but the frame-test space of a Hilbert space is not; neither, in general, is $A^{\#}$, even if A is projective. The following is related to Theorem 117 in [15].

Lemma 5: *Let (A, σ) be a #-algebra. The following are equivalent:*

(a) A is cohesive and projective

(b) A is cohesive and for all $a, b \in \mathcal{E}(A)$,

$$a \sim b \Rightarrow \sigma(a) = \sigma(b) \tag{2}$$

(c) A is algebraic and projective

Proof: The only nontrivial part is (b) $\Rightarrow$ (c). If $a, b, d \in \mathcal{E}(A)$ with $a \sim b$ and b co d, then $\sigma(a) = \sigma(b)$ by (2). Since any coherence is test preserving, $\sigma(b)$ co $\sigma(d)$, whence, $\sigma(a)$ co $\sigma(d)$, whence $\{\sigma(a), \sigma(d)\} \in \mathcal{M}(A)$. Since $a \sim \sigma(a)$ and σ is a cohesion, $a \cup \{\sigma(d)\} \in \mathcal{M}(A)$. Since $\{\sigma(d)\} \sim d$, the fact that σ is a cohesion also gives us $a \cup d \in \mathcal{M}(A)$. Thus, $\mathcal{M}(A)$ is algebraic. To see that it's projective, observe that if $x, y \in X(A)$ with $\{x\} \sim \{y\}$, then $x = \sigma(\{x\}) = \sigma(\{y\}) = y$. $\square$.

Cohesions and Quantum Logic

Two further conditions on a test space that are important in connecting test spaces to orthomodular structures are *coherence* and *regularity*. If $\mathcal{M}$ is a test space and $a \subseteq \bigcup \mathcal{M}$, let $a^{\perp}$ denote the set of outcomes orthgonal to every outcome in a. $\mathcal{M}$ is said to be coherent iff, for all events $a, b \in \mathcal{E}(\mathcal{M})$, $a \subseteq b^{\perp}$ implies $a \perp b$, and regular iff $a \sim b$ implies $a^{\perp} = b^{\perp}$. A coherent test space is regular iff it is algebraic, and the logic of a coherent, regular test space is a complete orthomodular poset (OMP) [15].

Definition: A model A is *unital* if, for ever $x \in X(A)$, there exists a state $\alpha \in \Omega(A)$ with $\alpha(x) = 1$, and *strongly unital* if for every pair of outcomes $x, y \in X(A)$ with $x \not\perp y$, there exits a state $\alpha \in \Omega(A)$ with $\alpha(x) = 1$ and $\alpha(y) > 0$.

Note that if A is strongly unital, it is also projective, since $x \sim y$ implies $x \not\perp y$.

Lemma 6: *Let A be strongly unital. If A has a cohesion, then $\mathcal{M}(A)$ is coherent and regular, hence algebraic, and $\Pi(A)$ is a complete OMP.*

Proof: Let $x \in a^{\perp}$. Then for every $\alpha \in \Omega(A)$ with $\alpha(x) = 1$, $\alpha(y) = 0$ for every $y \in a$, and hence, $\alpha(a) = 0$. Thus, as $\sigma(a) \sim a$, $\alpha(\sigma(a)) = 0$. From strong unitality, it follows that $x \perp \sigma(a)$, whence, since σ is a cohesion, we have $x \perp a$. Thus, $\mathcal{M}(A)$ is coherent. By Lemma 5, $\mathcal{M}(A)$ is algebraic, so it follows that $\mathcal{M}(A)$ is also regular. $\square$

5 The Compounding Monad

Given two models A and B, a simple model for a two-stage sequential experiment is as follows. Let $E \in \mathcal{M}(A)$; for each outcome $x \in E$, select a test $F_x \in \mathcal{M}(B)$. Perform the test E; upon obtaining outcome x, perform the pre-selected test F_x. If this yields outcome y, record the pair (x, y) as the outcome of the two-stage test. The outcome set for this experiment is then $\bigcup_{x \in E} \{x\} \times F_x$. We can define a model $\overrightarrow{AB}$, the *forward product*, of A and B [8] as follows: $\mathcal{M}(\overrightarrow{AB})$ is the collection of all such two-stage tests Note that then $X(\overrightarrow{AB}) = X(A) \times X(B)$. Probability weights on $\mathcal{M}(\overrightarrow{AB})$ are uniquely defined by pairs (α, β) where $\alpha \in \Pr(\mathcal{M}(A))$ and $\beta \in \Pr(\mathcal{M}(B))^{X(A)}$ via the formula

$$(\alpha; \beta)(x, y) := \alpha(x)\beta_x(y).$$

We take $\Omega(\overrightarrow{AB})$ to consist of those weights $(\alpha; \beta)$ with $\alpha \in \Omega(A)$ and $\beta_x \in \Omega(B)$ for all $x \in X(A)$.

One can enlarge a model A to obtain a model, A^c, the *compounding* of A, that is closed under the formation of sequential measurements. For test spaces, the construction is due to Foulis and Randall [5, 4], and the extension to arbitrary models is straightforward. The outcome-set $X(A^c)$ is the free monoid $X(A)^*$ on $X(A)$ with identity element e (the empty string). We identify $X(A)$ with the subset of $X(A)^*$ consisting of length-one strings. $\mathcal{M}(A^c)$ is the smallest collection $\mathcal{D}$ of subsets of $X(A)^*$ containing $\mathcal{M}(A)$ and closed under the formation of sets of the form $\bigcup_{x \in E} xF_x$ where $E \in \mathcal{D}$ and, for every $x \in E$, $F_x \in \mathcal{M}(A)$. By construction, $\mathcal{M}(A) \subseteq \mathcal{M}(A^c)$. Every set $E \in \mathcal{M}(A^c)$ consists of reduced words of some bounded length (since sets in $\mathcal{M}(A)$ have length 1 and the collection of sets bounded in this way has the required closure property).

To complete the description of the model A^c, we need to specify the states. Probability weights on $\mathcal{M}(A^c)$ are associated with *transition functions* $f : X^* \times X \to [0,1]$: one defines a probability weight ω_f recursively by $\omega_f(e) = 1$, $\omega_f(ax) = \omega_f(a)f(a,x)$. We say that ω_f is a *state* of A^c iff $f(a,x) \in \Omega(A)$ for every $a \in X^*$.

It is not hard to check that if A is unital, strongly unital, or algebraic, then so too is A^c. A morphism $\phi : A \to B$ naturally extends to a morphism $c(\phi) = \phi^c : A^c \to B^c$, given by

$$\phi^c(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n)$$

with $\phi^c(e) = e$. Note that if $\omega = \omega_f \in \Omega(B^c)$, then

$$\phi^{c*}(\omega_f)(ax) = \omega_f(\phi^c(a)\phi(x)) = \omega_f(\phi^c(a))f(\phi^c(a), \phi(x)).$$

Arguing inductively, the latter is in $\Omega(A^c)$.

Thus, $(\ \cdot\ )^c$ is an endofunctor on **Prob**. The natural semigroup homomorphisms $X^{**} \to X^*$ (concatenation) and $X \subseteq X^*$ define morphisms in **Prob**, and, since they also make $(\ \cdot\ )^*$ a monad in **Set**, we see that $(\ \cdot\ )^c$ is a monad in **Prob**.

Sequential Models Algebras for the compounding monad are easily characterized.

Definition: Let X be a monoid. A family of sets $\mathcal{A} \subseteq 2^X$ is *inductive* iff, for every $E \in \mathcal{A}$ and every function $F : E \to \mathcal{A}$, the set

$$\bigcup_{x \in E} xF_x = \{xy | x \in E, y \in F_x\}$$

belongs to $\mathcal{A}$.

A *sequential probabilistic model* is a model A such that $X(A)$ is a monoid, $\mathcal{M}(A)$ is inductive, and $\Omega(A)$ is *closed under conditioning.* This last means that if $\omega \in \Omega(A)$, $x \in X(A)$ with $\omega(x) > 0$, then $y \mapsto \omega(xy)/\omega(x)$

Proposition 2: *Sequential models are exactly the* $(\ \cdot\ )^c$ *algebras*

Proof: Let A be a sequential model and let $\phi : X(A)^* \to X(A)$ be the canonical mapping taking a string of elements of $X(A)$ to their product. This will define a $(\ \cdot\ )^c$ structure if it is a morphism. Since μ is the identity on $X(A)$ (understood as a subset of $X(A)^*$), μ carries $\mathcal{M}(A) \subseteq \mathcal{M}(A^c)$ to itself. Since $\mathcal{M}(A)$ is inductive in $\mathcal{P}(X(A))$, its preimage under μ is an inductive set containing $\mathcal{M}(A)$, and thus, contains $\mathcal{M}(A^c)$. Thus, μ takes $\mathcal{M}(A^c)$ to $\mathcal{M}(A)$. Finally, if ω is a state in $\Omega(A)$, then for any $x \in X(A)$ with $\omega(x) > 0$, define $\beta_x(y) = \omega(xy)/\omega(x)$, noting that this belongs to $\Omega(A)$. We now have

$$\mu^*(\omega)(x,a) = \omega(x\mu(a)) = \omega(x)\beta_x(\mu(a)) = \omega(x)\mu^*(\beta_x)(a).$$

Arguing inductively, we now see that $\mu^*(\beta)$ belongs to $\Omega(A^c)$.

For the converse, suppose A is a $(\cdot)^c$ algebra, with unit and multiplication $\eta : A \to A^c$ and $\phi : A^c \to A$. Applying the functor $X : \mathbf{Prob} \to \mathbf{Set}$, we see that $X(A)$ is a monoid with product $x, y \mapsto \phi(xy)$ and identity element $\phi(1_{A^c})$. Since $\eta : A \to A^c$ is an embedding and $\phi \circ \eta = \mathrm{id}_A$, we see that ϕ is test preserving, by Lemma 1. If $E \in \mathcal{M}(A)$ and $F : E \to \mathcal{M}(A)$, then we have $\bigcup_{x \in E} \{x\} \times F_x \in \mathcal{M}(A^c)$. The image of this test under ϕ is $\bigcup_{x \in E} xF_x$, so $\mathcal{M}(A)$ is inductive. $\square$

Remark: It is a perennial concern to try to define some satisfactory "sequential product" of effects, so that the product $a \star b$ of two effects can be read as "first a, and then b" [2, 12, 9, 14]. However, the standard candidate for quantum effects, $a \star b = \sqrt{a} b \sqrt{a}$, is not associative, raising difficulties for the intended interpretation [11]. The structure of A^c may help us better understand why the idea of forming a $\star$ product is inherently problematic. If $a, b \in \mathcal{E}(A)$ and $c \in \mathcal{E}(A^c)$, then if $a \sim b$, we have $ca \sim cb$, but in general $ac \not\sim bc$. Thus, if A is algebraic, we have a well-defined *action* of $\mathcal{E}(A^c)$ on $\Pi(A)$, namely $c[a] = [ca]$, but *not* a well-defined product $[a], [c] \mapsto [ac]$. Similar remarks apply to the effect algebra $[0, u_A]$ associated with $\mathbb{E}(A)$: the monoid $\mathcal{E}(A^c)$ acts on the the effect algebra, but this action generally does not "linearize" in the first variable to give a sensible sequential product on the latter.

6 Closing under both # and $(\cdot)^c$.

Since both compounding and coarse-graining are operationally reasonable ways of synthesizing new experiments from existing ones, one would like to be able to close a given model under both constructions. We have two endo-functors on **Prob**, given on objects by by $F(A) = (A^{\#})^c$ and $G(A) = (A^c)^{\#}$. For any model A, there is a natural morphism

$$\ell_A : (A^{\#})^c \to (A^c)^{\#}$$

given by

$$(a_1 \cdots a_n) \mapsto a_1 \times \cdots \times a_n.$$

where the product on the right represents the setwise product in X^* of the sets $a_i \subseteq X^*$. This gives us a natural transformation from F to G, and is in fact a distributive law [1] between the monads # and $(\cdot)^c$. Equivalently, the coarse-graining monad lifts to the category of $(\cdot)^c$-algebras, i.e., sequential models: if (A, ν) is a sequential model, define a sequential product on $A^{\#}$ in the obvious way, that is, for nonempty events a and b, set $ab := \{xy | x \in a, y \in b\}$. A G-algebra, then, carries both a coherence σ and a sequential product $x, y \mapsto xy$, and these interact according to $\sigma(xa) = x\sigma(a)$ and $\sigma(a)x = \sigma(ax)$ for any outcome $x \in X(A)$ and any non-empty event $a \in \mathcal{E}(A)$.

When A is algebraic, projective, and strongly unital, the logic $L = \Pi(A)$ will be a complete OMP, equipped with an associative binary operation distributing over orthogonal joins. But even absent these constraints, G-algebras support a notion of interference. As mentioned earlier, if $a, b \in \mathcal{E}(A)$ with $a \sim b$, it needn't be the case that $ay \sim by$ for a given outcome $y \in X(A)$. In particular, if $\omega \in \Omega(\overrightarrow{AB})$, the probabilities $\omega(\sigma(a)y)$ and $\omega(ay) = \sum_{x \in a} \omega(x, y)$ may differ. This is exactly what we see in quantum mechanical experiments. Borrowing language from quantum theory, we may say that ω exhibits "interference" between the outcomes $x \in a$ [17].

7 Conclusion and Prospectus

As we've seen, the simple requirement that a probabilistic model be closed under both coarse-graining and the formation of sequential experiments leads to rich and interesting structures, namely G-algebras. Imposing the modest further conditions of projectivity and strong unitality, one ends up essentially recovering complete OMPs — the traditional models of quantum logics — equipped with a kind of sequential product.

Looking ahead, The following questions seem especially important.

Question 1: If A is cohesive and projective, then A is algebraic, and hence, so is $G(A)$. What can be said about the detailed structure of $G(A)$ and its logic, $\Pi(G(A))$? The results of [4, 13] are likely to be relevant here.

Question 2: Which, if any, of the categories $\mathbf{Prob}^{\#}$, $\mathbf{Prob}^{c}$ and $\mathbf{Prob}^{G}$ of algebras for $\#$, $(\cdot)^c$, and G, admit a non-signaling monoidal structure such that, for some algebras A and B, AB will have entangled states?

Question 3: If A is a von Neumann model, what is the structure of $G(A)$? Is it embeddable in any von Neumann model?

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