Information-theoretic derivation of energy and speed bounds

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Information-theoretic insights have proven fruitful in many areas of quantum physics. But can the fundamental dynamics of quantum systems be derived from purely information-theoretic principles, without resorting to Hilbert space structures such as unitary evolution and self-adjoint observables? Here we provide a model where the dynamics originates from a condition of informational non-equilibrium, the deviation of the system's state from a reference state associated to a field of identically prepared systems. Combining this idea with three basic information-theoretic principles, we derive a notion of energy that captures the main features of energy in quantum theory: it is observable, bounded from below, invariant under time-evolution, in one-to-one correspondence with the generator of the dynamics, and quantitatively related to the speed of state changes. Our results provide an information-theoretic reconstruction of the Mandelstam-Tamm bound on the speed of quantum evolutions, establishing a bridge between dynamical and information-theoretic notions.

Introduction. Information-theoretic notions play an important role in modern physics, from thermodynamics to quantum field theory and gravity. With the advent of quantum information and computation, new links between information-theoretic primitives and physical laws have been uncovered, suggesting that information could be the key to understand the counterintuitive laws of quantum mechanics [1, 2]. Over the past two decades, this research program resulted into a series of reconstructions of the quantum framework from information-theoretic principles [3–9].

A limitation of most quantum reconstructions, however, is that they do not provide direct insights into the dynamics of quantum systems. Crucially, they do not provide an information-theoretic characterization of the notion of energy, with its dual role of observable quantity and generator of the dynamics. This observable-generator duality has been long known to be a fundamental feature, from which much of the algebraic structure of quantum theory can be derived [10]. Recent works assumed the duality as an axiom for the reconstruction of quantum theory [9], or postulated it as a requirement for constructing post-quantum dynamics [11–13]. And yet, little is known about the origin of this duality. How is "energy" emerging from "information," and why does it drive the evolution of quantum systems?

Here, we provide an information-theoretic characterization of quantum dynamics, deriving energy and the observable-generator duality from basic principles about the exchange of information in elementary physical systems. We use the framework of general probabilistic theories (GPTs) to model the dynamics as a sequence of elementary collisions through which a target system interacts with a field of identically prepared systems, as illustrated in Figure 1. The evolution takes place when the system's state deviates from the reference state of the field, a condition that we name *informational non-equilibrium.*

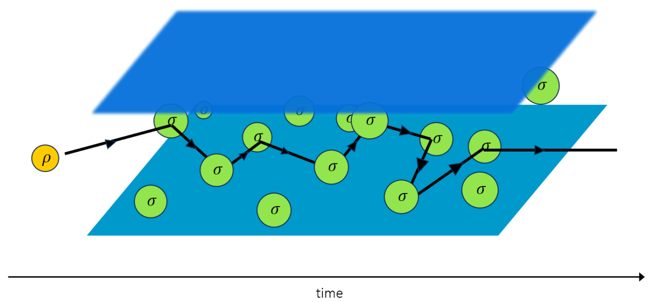

FIG. 1. Dynamics as a sequence of elementary collisions. A target system in an initial state ρ interacts with a field of independent and identically prepared systems, each of them in the reference state σ. Non-trivial dynamics takes place when the state ρ deviates from the reference state σ, a condition referred to as informational non-equilibrium.

We then show that the evolution induced by collisions converges to a reversible dynamics in the continuous-time limit, and that the resulting dynamics is canonically associated to an observable, which we interpret as the energy observable. This canonical energy observable shares most of the relevant features of energy in quantum mechanics: it is bounded from below, is a constant of motion, and is in one-to-one correspondence with the generator of the dynamics. Building on the notion of canonical energy observable, we then provide an information-theoretic derivation of the Mandelstam-Tamm bound on the speed of quantum evolutions [14], thus establishing a link between information-theoretic principles and dynamical notions such as energy and speed.

Collision models in GPTs. Here we formulate a general notion of collision model in the operational framework of GPTs, where the states of physical systems are not necessarily density matrices, but rather elements of some general convex sets (see the Supplemental Material [15] for details). Collision models have been widely studied in the theory of quantum open systems [16–22], with applications to non-equilibrium quantum thermodynamics [23, 24] and quantum machine learning [25]. In a collision model, a target system evolves through a sequence of quick pairwise interactions (collisions) with a large number of independent and identically prepared systems, as depicted in Fig. 1. Each interaction results into a joint reversible dynamics of the target system (denoted by A) and one of the other systems (denoted by A′). As a result of the interaction, the target system will evolve according to the effective evolution

$$\xrightarrow{\mathrm{A}}\boxed{\mathcal{C}_{\tau,\sigma}}\xrightarrow{\mathrm{A}} \;:=\; \begin{array}{c} \xrightarrow{\mathrm{A}} \\ (\sigma\xrightarrow{\mathrm{A}'} \end{array}\boxed{\mathcal{S}_\tau}\begin{array}{c} \xrightarrow{\mathrm{A}} \\ \xrightarrow{\mathrm{A}'} u) \end{array}, \qquad (1)$$

where $\mathcal{S}_\tau$ is the joint reversible dynamics of systems A and A′ resulting from a collision of interaction time τ, σ is the initial state of A′, hereafter called the *reference state*, and u is the operation of discarding system A′, mathematically represented by the functional that evaluates to 1 on every state of system A′ (see *e.g.* [26]).

The single-collision time τ is typically taken to be short compared to the overall evolution time t, while the total number of collisions N is taken to be large, with $N \propto t/\tau$. In the continuous limit, the evolution that system A is given by the transformation $\mathcal{U}_{t,\sigma} := \lim_{\tau\to 0}\left(\mathcal{C}_{\tau,\sigma}\right)^{\frac{\gamma t}{\tau}}$, where $\gamma > 0$ is a suitable constant such that $N = \gamma t/\tau$.

We call the evolution $\mathcal{U}_{t,\sigma}$ the *collisional dynamics* generated by the reference state σ. Our first result is an explicit expression for the collisional dynamics:

Theorem 1. *For every reference state σ, one has $\mathcal{U}_{t,\sigma} = e^{G_\sigma t}$, where G_σ is the linear map uniquely defined by the relation $G_\sigma(\rho) := (\mathcal{I}_A \otimes u_{A'}) \frac{\mathrm{d}\mathcal{S}_\tau}{\mathrm{d}\tau}\big|_{\tau=0}(\rho\otimes\sigma)$ for every state ρ, and $\mathcal{I}_A$ is the identity map on the states of system A.*

The proof is provided in the Supplemental Material [15].

An important consequence of theorem 1 is that the collisional dynamics $\mathcal{U}_{t,\sigma}$ is reversible for every σ and for every t. Ideally, the evolution can be reversed by replacing τ by $-\tau$ in Eq. (1), while keeping the state σ fixed. This approach corresponds to an inversion of the joint time evolution during each collision. Later in the paper, we will also see that, under a few assumptions, the inverse can also be realized by replacing the state σ with a new state $\widetilde{\sigma}$ in Eq. (1), while keeping the interaction time τ positive.

Informational equilibrium. So far, the formulation of the collision model did not require any assumption, other than the basic framework of general probabilistic theories. We now explore the idea that dynamics could be linked to deviations of the system's state from the state of the surrounding environment. To this purpose, we take the systems A′ in the collision model to be of the same type of the target system A. When the states ρ and σ coincide, we say that the system is at *informational equilibrium* with its environment. Our key assumption, hereafter denoted as IE, is that at informational equilibrium, the target and ancillary system do not undergo any evolution, namely

$$\mathcal{S}_\tau(\rho\otimes\rho) = \rho\otimes\rho \qquad \forall\tau\in\mathbb{R}\,. \qquad (2)$$

In short: *information equilibrium prevents evolution.*

In the following we show that Eq. (2), combined with three common assumptions in the GPT framework, gives rise to a notion of energy that captures many of the features of energy in quantum mechanics.

The first assumption is *purity preservation (PP)* [27], namely the requirement that composing two pure transformations in parallel or in sequence gives rise to another pure transformation (informally, a pure transformation is one that cannot be viewed as the coarse-graining of a set of transformations; see Ref. [15] for the precise definition). This assumption happens to be satisfied by all concrete examples of GPTs known to date, with the notable exception of Ref. [28].

The second assumption is *strong symmetry (SS)* [9], the condition that every two sets of perfectly distinguishable states can be reversibly converted into one another if they have the same cardinality. Strong symmetry has a clear information-theoretic interpretation, as it guarantees that two sets that can perfectly encode the same amount of information are equivalent. Here, we assume that strong symmetry holds for the collisional dynamics.

Finally, the third assumption is classical decomposability [9], also called *diagonalization (D)* [29], namely the requirement that every state can be prepared as a random mixture of perfectly distinguishable pure states. Diagonalization and strong symmetry appear either as axioms or as key results in information-theoretic derivations of quantum theory [3–7, 9] and often play a role in the study of information processing in other examples of GPTs [27, 30–33].

Information-theoretic characterization of energy. Theorem 1 sets up a correspondence between states and generators of the dynamics via the fast collision model.

Theorem 2. *For any collision model, IE, SS, and D imply that the correspondence $\sigma \mapsto G_\sigma$ between states and generators is injective: if two states generate the same collisional dynamics, then they are the same state.*

The proof is provided in Supplemental Material [15], where we show that the theorem still holds if SS is replaced by the weaker assumption that every two pure states are connected by a reversible transformation generated by the collision model.

Theorem 2 establishes a one-to-one correspondence between states and generators of the collisional dynamics. Now, earlier work by Barnum, Müller, and Ududec [9] showed that SS and D imply a one-to-one correspondence between states and *effects*, that is, probability functionals associated to measurement outcomes (see the next paragraph for more details). Thanks to this fact, the state-generator correspondence of Theorem 2 can be turned into an effect-generator correspondence.

Effects are associated to the outcomes of possible measurements, and constitute the primary observable quantities in a physical theory. Mathematically, an N-outcome measurement on system A is described by a collection of functionals $(e_i)_{i=1}^N$, mapping states ρ of system A into probabilities $e_i(\rho) \in [0,1]$ and satisfying the normalization condition $\sum_{i=1}^N e_i(\rho) = 1$ for every possible state ρ (see *e.g.* [26]). Operationally, $e_i(\rho)$ is the probability that the measurement produces outcome i when performed on a system in the state ρ. An effect in a GPT is a functional appearing in one of the possible measurements allowed by the theory. In the following, we will denote by $\mathsf{St}(\mathrm{A})$ and $\mathsf{Eff}(\mathrm{A})$ the sets of states and effects for system A, respectively.

SS and D imply that every every pure state $\psi \in \mathsf{St}(\mathrm{A})$ is uniquely associated to a pure effect $e_\psi \in \mathsf{Eff}(\mathrm{A})$ satisfying the condition $e_\psi(\psi) = 1$ [9]. Mathematically, the map $\psi \mapsto e_\psi$ is a one-to-one correspondence between the set of pure states and the set of pure normalized effects, that is, pure effects e such that $e(\rho) = 1$ for some state ρ. This correspondence can be extended by linearity to mixed states, and, more generally to arbitrary linear combinations of pure states, giving rise a property known as (strong) self-duality [34, 35].

We now use self-duality to define a canonical energy observable. An observable on system A can be defined as an element X of the real vector space spanned by the effects $\mathsf{Eff}(\mathrm{A})$ [27]. For a linear combination of effects $\{e_j\}$ with coefficients $\{x_j\}$, the expectation value of the observable $X = \sum_j x_j\, e_j$ on the state ρ is given by $\langle X\rangle_\rho := X(\rho) = \sum_j x_j\, e_j(\rho)$, and can be estimated by performing suitable measurements containing the effects $\{e_j\}$ and by post-processing the outcomes.

A notion of energy observable should capture two key aspects of the dynamics: (1) the trajectories for all possible initial states, and (2) the rate at which such trajectories are travelled. Referring to the basic scheme of the collision model [Eq. (1)], the state space trajectories are uniquely determined by the reference state σ, and therefore, by the effect e_σ corresponding to σ through self-duality. As a quantifier of the rate, here we take the maximum singular value of the generator $G_\sigma = \frac{\mathrm{d}\mathcal{U}_{t,\sigma}}{\mathrm{d}t}\big|_{t=0}$, which corresponds to the maximum rate for the change of vectors under the linear map $\mathcal{U}_{t,\sigma} = e^{G_\sigma t}$ (see Supplemental Material [15] for details). These considerations lead to the following definition:

Definition 1. *In a GPT satisfying IE and self-duality, the canonical energy observable associated to the generator G_σ is the functional $H = \lambda_{\max}(G_\sigma)\, e_\sigma$, where e_σ is the effect associated to the state σ, and $\lambda_{\max}(G_\sigma)$ is the maximum singular value of the generator G_σ.*

Note that the canonical energy observable is automatically bounded from below, since one has $\langle H\rangle_\rho \geq 0$ for every state ρ.

In a theory satisfying SS and D, the energy observable H can be measured in a canonical way, by performing an ideal measurement described by pure effects. Explicitly, the canonical measurement can be obtained by diagonalizing the state σ as $\sigma = \sum_{i=1}^d p_i\, \psi_i$, where $\{p_i\}_{i=1}^d$ are probabilities and $\{\psi_i\}_{i=1}^d$ is a maximal set of perfectly distinguishable pure states. Combining this decomposition with self-duality, the energy observable can be written as

$$H = \sum_{i=1}^d E_i\, e_{\psi_i} \qquad E_i := \lambda_{\max}(G_\sigma)\, p_i\,. \tag{3}$$

Here, the pure effects $\{e_{\psi_i}\}$ form a measurement [15], which we interpret as the ideal energy measurement, with outcomes $\{1, \dots, d\}$ associated to the possible energy values $\{E_1, \dots, E_d\}$, respectively. Eq. (3) gives a canonical way to estimate the expectation value of the energy for every possible input state.

Crucially, the canonical energy observable is invariant under time evolution. The invariance of H follows from the invariance of the state σ, which in turn is a direct consequence of informational equilibrium: if a system in the state σ collides with another system in the state σ, then no state change occurs to them due to Eq. (2). By strong self-duality, the invariance of σ implies the invariance of e_σ, and therefore of H. Physically, this condition amounts to the conservation of the energy: for every state ρ, the expectation value $\langle H\rangle_{\rho_t}$ is constant along the trajectory $\rho_t := \mathcal{U}_{t,\sigma}(\rho)$, $t \in \mathbb{R}$. In the Supplemental Material [15], we show that, for theories satisfying IE, SS, D, and PP, not only the expectation value, but also the whole probability distribution of the ideal energy measurement is invariant under time evolution.

Speed bound. Using the canonical energy observable, we now derive a fundamental speed limit, which in quantum theory coincides with the Mandelstam-Tamm bound [14]. Our limit provides a lower bound, expressed in terms of the variance of the energy observable, on the time taken by a system to transition from a given initial state to a given final state.

We start by providing a notion of speed in general probabilistic theories:

Definition 2 (Speed of state change)**.** *Let $\mathcal{D}_t$ be a dynamics, let $\{\rho_t = \mathcal{D}_t\rho \mid t \in \mathbb{R}\}$ be the trajectory of an initial state ρ, and let t_0 and $t_1 \geq t_0$ be two moments of time. The average speed from time t_0 to time t_1 is defined as*

$$v_\rho(t_0, t_1) := \frac{\|\rho_{t_1} - \rho_{t_0}\|}{t_1 - t_0}\,, \tag{4}$$

where $\|\cdot\|$ an arbitrary norm on the real vector space generated by the states of A. Similarly, the instantaneous speed at time t is defined as $v_\rho(t) = \lim_{\delta t\to 0} v_\rho(t, t+\delta t)$.

In the following we will take the norm $\|\cdot\|$ to be the the Euclidean norm induced by self-duality, namely $\|\sum_j c_j\,\rho_j\| := \sqrt{\sum_{j,k} c_j, c_k e_{\rho_j}(\rho_k)}$ for every linear combination of states (ρ_j) with real coefficients (c_j). With this choice of norm, the instantaneous speed is constant along the trajectory for every reversible dynamics with time-independent generator $\mathcal{U}_t = e^{Gt}$. Using this fact, in the Supplemental Material [15] we prove the inequality

$$v_\rho(t_0, t_1) \leq v_\rho(t) = \|G\,\rho\| \qquad \forall t_0, t_1, t \in \mathbb{R}\,, \tag{5}$$

which we use to derive a bound on the time taken by the system to transition between two given states:

Theorem 3 (Speed bound)**.** *In a GPT satisfying IE, PP, SS, and D, the time Δt taken by a system to transition from an initial state ρ_0 to a final state ρ_1 through a collisional dynamics is lower bounded as*

$$\Delta t \geq \frac{D(\rho_0, \rho_1)}{\Delta H}\,, \tag{6}$$

where $D(\rho_0, \rho_1) := \|\rho_1 - \rho_0\|/\sqrt{2}$ is the normalized Euclidean distance between the states ρ_0 and ρ_1, while $\Delta H := \sqrt{\langle H^2\rangle_{\rho_0} - \langle H\rangle^2_{\rho_0}}$ is the standard deviation of the canonical energy observable H, and H^2 is the observable defined as $H^2 := \sum_i E_i^2\ e_{\psi_i}$.

The speed limit (6) provides a general lower bound that applies to all possible collisional dynamics and to all possible pairs of states. For two perfectly distinguishable pure states, the bound becomes $\Delta t \geq 1/\Delta E$, which in the case of quantum theory coincides with the Mandelstam-Tam bound up to a dimensional factor $h/4$, where h is Planck's constant [14]. Hence, Eq. (6) can be regarded as an alternative, purely information-theoretic derivation of the Mandelstam-Tamm bound.

The speed of the inverse evolution. Mathematically, inverting a dynamics amounts to changing its generator G into $-G$. In terms of the collision model, this would amount to changing the reference state σ into $-\sigma$, which however is not a state. Hence, a natural question is: how to achieve the inverse evolution in the collision model? We now show that there exists a valid state $\widetilde{\sigma}$, called the *inverting state*, such that $G_{\widetilde{\sigma}}$ is proportional to $G_{-\sigma}$. For a reference state of the form $\sigma = \sum_{i=1}^d p_i\,\psi_i$, $\{\psi_i\}$ being a maximal set of perfectly distinguishable pure states, the inverting state is $\widetilde{\sigma} = \sum_i (p_{\max} - p_i)\,\psi_i/(d\,p_{\max} - 1)$ with $p_{\max} := \max\{p_i\}_{i=1}^d$. In the Supplemental Material, we show that the generator associated to this state is $G_{\widetilde{\sigma}} = -G_\sigma/(d\,p_{\max} - 1)$. In words, $G_{\widetilde{\sigma}}$ is proportional to $-G_\sigma$, the generator of the inverse evolution. Crucially, a consequence of the proportionality factor is that the time required to achieve the direct evolution $\mathcal{U}_t = e^{G_\sigma t}$ is generally different from the time required to achieve its inverse $\mathcal{U}_t^{-1} = e^{-G_\sigma t}$. The two times are related through the equation

$$t_{\text{inverse}} = (d\,p_{\max} - 1)\,t_{\text{direct}}\,. \tag{7}$$

For example, consider the case where the direct dynamics is generated by a pure state ψ, corresponding to $p_{\max} = 1$. In this case, the time required by the inverse evolution is $d - 1$ times the time required for the inverse evolution. For physical systems consisting of a large number of components, d grows exponentially and therefore the inverse evolution requires an exponentially larger time. It is intriguing to speculate that, if the collision model were taken to be fundamental (meaning that the natural dynamics of physical systems does indeed arise from interactions with a background field of identically prepared particles), the above argument could provide an explanation for why certain evolutions are physically hard to invert.

Conclusions. In this work we have defined a canonical notion of energy observable from information-theoretic principles and derived its duality with the generators of dynamics in collision models. Our results provide a principled approach to the understanding of energy in quantum theory, by placing it into the broader context of general probabilistic theories. An important byproduct of our analysis is a derivation of a speed limit on the amount of time needed to transition between two states through a collisional dynamics. In the case of quantum theory, our bound reduces to the well-known Mandelstam-Tamm bound, sometimes referred to as form of "energy-time uncertainty relation" [36]. More generally, our bound can be used to derive generalized uncertainty relations for arbitrary one-parameter groups of reversible transformations, without necessarily regarding the group parameter as "time." These results contribute with new insights to a recently line of investigation on various types of uncertainty relations in general probabilistic theories [37–40].

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