CP$^\infty$ and beyond: 2-categorical dilation theory

https://arxiv.org/pdf/2310.15776.pdf

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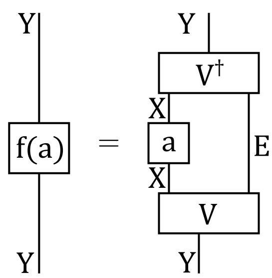
June 14, 2024

1 Algebraic quantum mechanics

Classical mechanics can be approached by considering states (positions, momenta) in a locally compact Hausdorff space X, with observables in the commutative von Neumann algebra $C(X)$ of continuous functions on X and evolution given by homeomorphisms from $C(X)$ to itself. To move to the probabilistic setting of statistical mechanics, one can take the convex set of unital completely positive (CP) maps from $C(X)$ to itself; these are the dynamical maps governing stochastic evolution. Similarly, quantum theory can be presented as states in a Hilbert space H, with observables in the von Neumann algebra $B(H)$ of bounded linear functions on H and evolution given by unitary operators from $B(H)$ to itself. To obtain the most general stochastic dynamics, we consider normal (i.e. weak $*$-continuous) unital CP maps on $B(H)$.

In general, one can identify physical systems with von Neumann algebras of observables on those systems, while the most general dynamical maps between systems are identified with normal unital CP maps (channels) between these observable algebras. This algebraic approach includes not only quantum-to-quantum, but also classical-to-quantum, quantum-to-classical and classical-to-classical dynamics [Hol03, §3.2].

The Hilbert space formulation of quantum theory is recovered from the von Neumann algebraic formulation by a representation result for quantum channels called *Stinespring's theorem*, which we now state in an appealing form given in [CH16, Cor. 12]. Any completely positive map $f : B(X) \to B(Y)$ between von Neumann algebras of bounded operators on Hilbert spaces can be written as conjugation by an bounded linear map $V : Y \to X \otimes E$, for some Hilbert space E (the environment). As a string diagram in the W^*-tensor category Hilb of Hilbert spaces and bounded linear maps (we read diagrams from bottom to top):



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We call the pair (E, V) a *representation* of the completely positive map f. The CP map f is unital if and only if V is an isometry, and any two Stinespring representations of the same map are related by a partial isometry on the environment system. The limitation of this formulation of Stinespring's theorem is that it only applies to maps between von Neumann algebras of all bounded operators on a Hilbert space.

Here we present a similar theorem for completely positive maps between arbitrary von Neumann algebras, using the W^*-2-category of von Neumann algebras, bimodules and intertwiners.

2 Categorical quantum mechanics

This work is inspired by the programme of categorical quantum mechanics, which studied quantum theory from the perspective of the W^*-tensor category Hilb of Hilbert spaces and linear maps. One goal of categorical quantum mechanics was to recover von Neumann algebras and channels via a categorical construction from Hilb.

In the case of finite-dimensional von Neumann algebras, this is possible using Frobenius algebras. A special symmetric Frobenius algebra in Hilb corresponds precisely to a finite-dimensional von Neumann algebra equipped with its canonical special trace [Vic11]. Furthermore, CP maps between finite-dimensional von Neumann algebras can be identified with morphisms between the corresponding Frobenius algebras obeying a certain positivity condition [CHK16] [HV19, §7.2.1]. The algebraic theory can therefore be recovered as a theory of Frobenius algebras in Hilb, in the finite-dimensional case.

Unfortunately, this does not generalise to infinite-dimensional von Neumann algebras, since Frobenius algebras, being self-dual, are necessarily finite-dimensional. To resolve this problem an alternative construction, the CP^∞ *construction*, was proposed [CH16]. The objects of the category $\mathrm{CP}^\infty(\mathrm{Hilb})$ are objects of Hilb, while morphisms $X \to Y$ are equivalence classes of isometries $Y \to X \otimes E$ in Hilb, where two isometries are equivalent iff they induce the same map $B(X) \to B(Y)$ by the conjugation (1). By the formulation of Stinespring's theorem stated above, this construction recovers the category of von Neumann algebras $B(H)$ of all bounded operators on a Hilbert space and channels between them. However, it does not recover the category of all von Neumann algebras and channels. We here present a 'horizontal categorification' of the CP^∞-construction, using the W^*-2-category of von Neumann algebras, bimodules and intertwiners, which does recover all von Neumann algebras and channels.

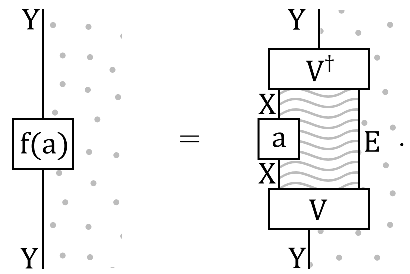
3 Our results

3.1 Stinespring's theorem

To achieve our generalisation of Stinespring's theorem to normal completely positive maps between arbitrary von Neumann algebras, we allow the CP map to be represented not just in the W^*-tensor category Hilb, but in the W^*-2-category $[W^*]$ of von Neumann algebras, bimodules and intertwiners [Lan00, §5]. This category has an equivalent, functorial definition as the 2-category of W^*-categories, normal unitary linear functors and bounded natural transformations (the equivalence is the usual one, which takes a von Neumann algebra to its W^*-category of right modules) [Yam07]. We will thereby be able to construct the category of all von Neumann algebras and channels by a natural generalisation of the CP^∞-construction on the W^*-2-category $[W^*]$.

A generating 1-morphism X is a 1-morphism for which every other 1-morphism of the same type can be realised as a subobject of a (possibly infinite) direct sum of copies of

X. Every von Neumann algebra is obtained as $\mathrm{End}(X)$ for some generating 1-morphism X in $[W^*]$ whose source is the trivial von Neumann algebra $\mathbb{C}$ (in fact, this extends to a classification of von Neumann algebras in terms of such generating 1-morphisms). Let $X : \mathbb{C} \to r$ and $Y : \mathbb{C} \to s$ be generating 1-morphisms. Then our generalised Stinespring's theorem is as follows: any normal completely positive map $f : \mathrm{End}(X) \to \mathrm{End}(Y)$ can be written as conjugation by a 2-morphism $V : Y \to X \otimes E$, for some 1-morphism $E : s \to r$ (the environment). As a string diagram in the W^*-2-category $[W^*]$:



Here we have left the regions corresponding to the object $\mathbb{C}$ unshaded, and the shaded regions correspond to the von Neumann algebras r (wavy lines) and s (dots), respectively.

Further, we prove that the CP map f is unital if and only if V is an isometry, f is a unital $*$-homomorphism iff V is unitary, and that different Stinespring representations of the same CP map are related by a partial isometry on the environment system. In fact, the different Stinespring representations of a given CP map form a dagger category. Every CP map has a minimal representation, which is an initial object in the dagger category of Stinespring representations; it is unique up to a unitary and is related to every other representation by a unique isometry.

3.2 The CP$^\infty$-construction

The 2-categorical generalisation of the CP$^\infty$ construction is now straightforward to state. Let CP be the category whose objects are von Neumann algebras and whose 1-morphisms are normal completely positive maps. Let Chan $\subset$ CP be the category whose objects are von Neumann algebras and whose objects are channels, i.e. unital normal completely positive maps.

The category CP can then be constructed from $[W^*]$, by taking objects to be generating 1-morphisms $X, Y, \ldots$ from $\mathbb{C}$, and morphisms to be equivalence classes of morphisms $Y \to X \otimes E$. The 2-category Chan can likewise be constructed from $[W^*]$ by restricting to equivalence classes of *isometric* 2-morphisms of type $Y \to X \otimes E$.

Note that we recover the original CP$^\infty$-construction if we restrict objects of CP to generating 1-morphisms of type $\mathbb{C} \to \mathbb{C}$ (since every object of the category Hilb is a generator). This is why our construction can be thought of as a horizontal categorification of the original CP$^\infty$-construction.

3.3 An application to extremal channels

The 2-categorical framework we have introduced has allowed us to construct the category of all von Neumann algebras and channels via the generalised CP$^\infty$-construction. To show that 2-categorical dilation theory is useful more generally, we use it to extend Choi's characterisation of extremal points in the convex set of channels between finite-dimensional matrix algebras [Cho75, Thm. 5] to extremal points in the convex set of channels between arbitrary von Neumann algebras. (C.f. [WW17, Thm. 32], which characterises extremal channels in terms of an injective affine order isomorphism, and [Moh18, Prop 4.1], which holds for channels whose target is of the form $B(H)$.)

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