Optimal protocols for universal adjointation of isometry operations

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Summary: Identification of possible transformations of quantum objects including quantum states and quantum operations is indispensable in developing quantum algorithms. Universal transformations, defined as input-independent transformations, appear in various quantum applications. Such is the case for universal transformations of unitary operations. However, extending these transformations to non-unitary operations is nontrivial and largely unresolved. Addressing this, we introduce *isometry adjointation* protocols that convert an input isometry operation into its adjoint operation, which include both unitary operation and quantum state transformations. The paper details the construction of parallel and sequential isometry adjointation protocols, derived from unitary inversion protocols using quantum combs, and achieving optimal approximation error. This error is shown to be independent of the output dimension of the isometry operation. In particular, we explicitly obtain an asymptotically optimal parallel protocol achieving an approximation error $\epsilon = \Theta(d^2/n)$, where d is the input dimension of the isometry operation and n is the number of calls of the isometry operation. The full paper of this work is on arXiv [1].

PROBLEM SETTING AND MAIN RESULTS

Quantum protocols dealing with unknown quantum states have been extensively studied, such as state cloning [2]. Possibility and impossibility of such protocols have played an important role in implementing cryptographic protocols [3, 4]. Unknown quantum operations are also utilized in various quantum protocols, such as oracle quantum computation [5], unitary property testing [6], and higher-order quantum transformations [7]. In general, it is difficult to utilize unknown quantum states and operations in quantum protocols since we require an extra resource overhead to estimate their description via process tomography [8, 9]. Previous works have invented subroutines to deal with unknown quantum states or unitary operations such as swap test [10], amplitude amplification [11], and transformations of unknown unitary operations [12–19]. However, their extension to general quantum operations are not well investigated. One of the most important class of quantum operations are isometry operations, which represent encoding of quantum information into a higher-dimensional system. Mathematically, they include unitary operations and pure quantum states as special cases, namely, $\mathbb{V}_{\mathrm{iso}}(d, D) \simeq \mathbb{U}(d)$ for $D = d$ and $\mathbb{V}_{\mathrm{iso}}(d, D) \simeq \mathbb{C}^D$ for $d = 1$ hold, where $\mathbb{V}_{\mathrm{iso}}(d, D)$ is the set of isometry operators $V : \mathbb{C}^d \to \mathbb{C}^D$ and $\mathbb{U}(d)$ is the set of d-dimensional unitary operators. In this work, we define the task *isometry adjointation* given as follows.

Definition 1 (Isometry adjointation)**.** *Given n calls of an unknown isometry operation $V_{\mathrm{in}} \in \mathbb{V}_{\mathrm{iso}}(d, D)$, the task is to implement a quantum instrument $\{\Phi_I, \Phi_O\}$[1] such that Φ_I approximates the adjoint operation $V_{\mathrm{in}}^\dagger$[2].*

The adjoint operation can be written as $V_{\mathrm{in}}^\dagger \rho_{\mathrm{in}} V_{\mathrm{in}} = \mathcal{V}_{\mathrm{in}}^{-1}(\Pi_{\mathrm{Im}\, V_{\mathrm{in}}} \rho_{\mathrm{in}} \Pi_{\mathrm{Im}\, V_{\mathrm{in}}})$, where $\mathcal{V}_{\mathrm{in}}^{-1}$ is a CPTP map satisfying $\mathcal{V}_{\mathrm{in}}^{-1} \circ \mathcal{V}_{\mathrm{in}} = \mathbb{1}_d$, and $\Pi_{\mathrm{Im}\, V_{\mathrm{in}}}$ is an orthogonal projector onto the image $\mathrm{Im}\, V_{\mathrm{in}}$. Thus, an isometry adjointation protocol checks whether the input quantum state is within the subspace $\mathrm{Im}\, V_{\mathrm{in}}$ specified by the unknown isometry operation V_{in}, and if the input state is in the subspace, it applies the inverse operation $\mathcal{V}_{\mathrm{in}}^{-1}$ on the input state (Fig. 1).

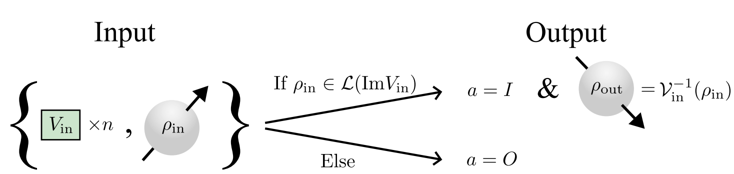

Figure 1. Definition of the task isometry adjointation.

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[1] The measurement outcomes I and O stand for "in $\mathrm{Im}\, V_{\mathrm{in}}$" and "out of $\mathrm{Im}\, V_{\mathrm{in}}$", respectively (see also Fig. 1).

[2] We also demand that the one-side error condition $\Phi_O \circ \mathcal{V}_{\mathrm{in}} = 0$, i.e., when $\rho_{\mathrm{in}} \in \mathcal{L}(\mathrm{Im}\, V_{\mathrm{in}})$, we obtain the measurement outcome $a = I$ with a unit probability.

This task reduces to unitary inversion [15–19] ($U \in \mathbb{U}(d) \mapsto U^{-1}$) for $D = d$ and swap test [10], or programmable projective measurement [20] ($|\psi\rangle \in \mathbb{C}^D \mapsto \{|\psi\rangle\langle\psi|, \mathbb{1} - |\psi\rangle\langle\psi|\}$) for $d = 1$. We show two ways to construct isometry adjointation protocols, one of which utilizes the input isometry operations in parallel, and the other utilizes them in sequence. The parallel protocol is constructed from a unitary estimation protocol, and the sequential protocol is from a unitary inversion protocol (Fig. 2). Both of them achieve the optimal performances among all parallel or sequential protocols.

Theorem 2. *The parallel or sequential protocols shown in Fig. 2 implement the quantum instrument* $\{\Phi_I, \Phi_O\}$ *satisfying*

$$\Phi_I(\rho_{\text{in}}) = (1-p)V_{\text{in}}^\dagger \rho_{\text{in}} V_{\text{in}} + \frac{\mathbb{1}_d}{d} \operatorname{Tr}\{[p\Pi_{\operatorname{Im} V_{\text{in}}} + \alpha(\mathbb{1}_D - \Pi_{\operatorname{Im} V_{\text{in}}})]\rho_{\text{in}}\}, \tag{1}$$

where $p, \alpha \in [0,1]$ *are obtained from the original unitary estimation or unitary inversion protocol. The worst-case diamond-distance error is given by* $\epsilon = \frac{1}{2}\sup_{V_{\text{in}} \in \mathbb{V}_{\text{iso}}(d,D)} \|\Phi - \mathcal{V}_{\text{adjoint}}\|_\diamond = \max\{\frac{1}{2}(1-d^{-2})p, \alpha\}$, *where* Φ *and* $\mathcal{V}_{\text{adjoint}}$ *are CPTP maps defined by* $\Phi \coloneqq \Phi_I \otimes |0\rangle\langle 0| + \Phi_O \otimes |1\rangle\langle 1|$ *and* $\mathcal{V}_{\text{adjoint}} \coloneqq \mathcal{V}^\dagger \otimes |0\rangle\langle 0| + \frac{\mathbb{1}_d}{d}\operatorname{Tr}[(\mathbb{1}_D - \Pi_{\operatorname{Im} V_{\text{in}}})\cdot] \otimes |1\rangle\langle 1|$.

Theorem 3. *For given* d, D, n, *the protocols shown in Figs. 2 achieve the optimal worst-case diamond-distance error among all parallel or sequential protocols, respectively.*

In particular, p, α in (1) is given in Theorems 5 and 6 of the technical manuscript [1], which do not depend on the output dimension D of the isometry. Thus, we obtain the following Lemma.

Lemma 4. *The optimal approximation error* ϵ *of parallel or sequential isometry adjoinattion using* n *calls of the input isometry operation* $V_{\text{in}} \in \mathbb{V}_{\text{iso}}(d, D)$ *do not depend on* D.

CONSTRUCTION OF PARALLEL AND SEQUENTIAL ISOMETRY ADJOINTATION PROTOCOLS

We construct a quantum instrument $\{\Psi_a : \mathcal{L}(\mathbb{C}^D)^{\otimes n+1} \to \mathcal{L}(\mathbb{C}^d)^{\otimes n+1}\}_{a\in\{I,O\}}$ using the quantum Schur transform [21–24] satisfying the following equation [25]:

$$\Psi_I[\mathcal{V}_{\text{in}}^{\otimes n}(\phi) \otimes \rho] = \int_{\mathbb{U}(d)} \mathrm{d}U \mathcal{U}^{\otimes n}(\phi) \otimes (\mathcal{U} \circ \mathcal{V}_{\text{in}}^\dagger)(\rho) + \operatorname{Tr}\left[(\mathbb{1}_D - \Pi_{\operatorname{Im} V_{\text{in}})})\rho\right]\Psi_I(\phi), \tag{2}$$

where $\mathcal{V}_{\text{in}}(\cdot) \coloneqq V_{\text{in}} \cdot V_{\text{in}}^\dagger$, $\mathcal{U}(\cdot) \coloneqq U \cdot U^\dagger$, $\mathrm{d}U$ is the Haar measure on $\mathbb{U}(d)$, and Ψ_I is a completely positive trace non-increasing (CPTNI) map. To cancel out $\mathcal{U}$ after $\mathcal{V}_{\text{in}}^\dagger$ in (2), a unitary estimation protocol is combined as shown in Fig. 2 (a). The left panel of Fig. 2 (a) shows a parallel protocol for unitary inversion using a unitary estimation protocol. The input unitary operation U_{in} is estimated as $\hat{U}_i$ from the measurement outcome i of a POVM $\{M_i\}$ on the state $\mathcal{U}_{\text{in}}^{\otimes n} \otimes \mathbb{1}(\phi)$. The inverse operation $R_i \coloneqq \hat{U}_i^{-1}$ of the estimated operation is applied to the input quantum state ρ_{in}. Assuming that the unitary estimation protocol is covariant, we can show that the quantum circuit shown in the right panel of Fig. 2 (a) implements a quantum operation (1) with $p = \frac{d^2}{d^2-1}(1 - F_{\text{est}})$ and $\alpha = \operatorname{Tr}\Psi_I[\operatorname{Tr}_{\mathcal{A}}(\phi)]$ where F_{est} is the entanglement fidelity of unitary estimation and ϕ is shown in Fig. 2. From covariant unitary estimation protocols presented in Refs. [26–29], we show the isometry adjointation protocol achieving $p = O(d^4/n^2)$, $\alpha = O(d^2/n)$ in (1). Thus, for $n \gg d^2$, this protocol achieves $\epsilon = O(d^2/n)$, and in particular for the case of $d = 2$, this is given by $\epsilon = 6.2287/n + O(n^{-2})$. Therefore, we can achieve an approximation error ϵ by $n = O(d^2/\epsilon)$ calls of the input isometry operation. We also show that this scaling is optimal among all possible parallel protocols, i.e., $\inf_{\text{parallel protocol}} \epsilon = \Theta(d^2/n)$.

From a given sequential unitary inversion protocol, we construct an isometry adjointation protocol as shown in Fig. 2 (b). This construction is done by inserting the set of quantum operations $\Gamma^{(i)}$ (red one) to the unitary inversion protocol composed of $\Lambda'^{(i)}$ (blue one). The sequence of $\Lambda'^{(i)}$ transforms the action of n calls of $\mathcal{V}_{\text{in}}$ to $\mathcal{V}_{\text{in}}^\dagger$ and randomized unitary operation, similarly to (2). We can show that the resulting protocol implements a quantum operation (1) if the original unitary inversion protocol is covariant [25].

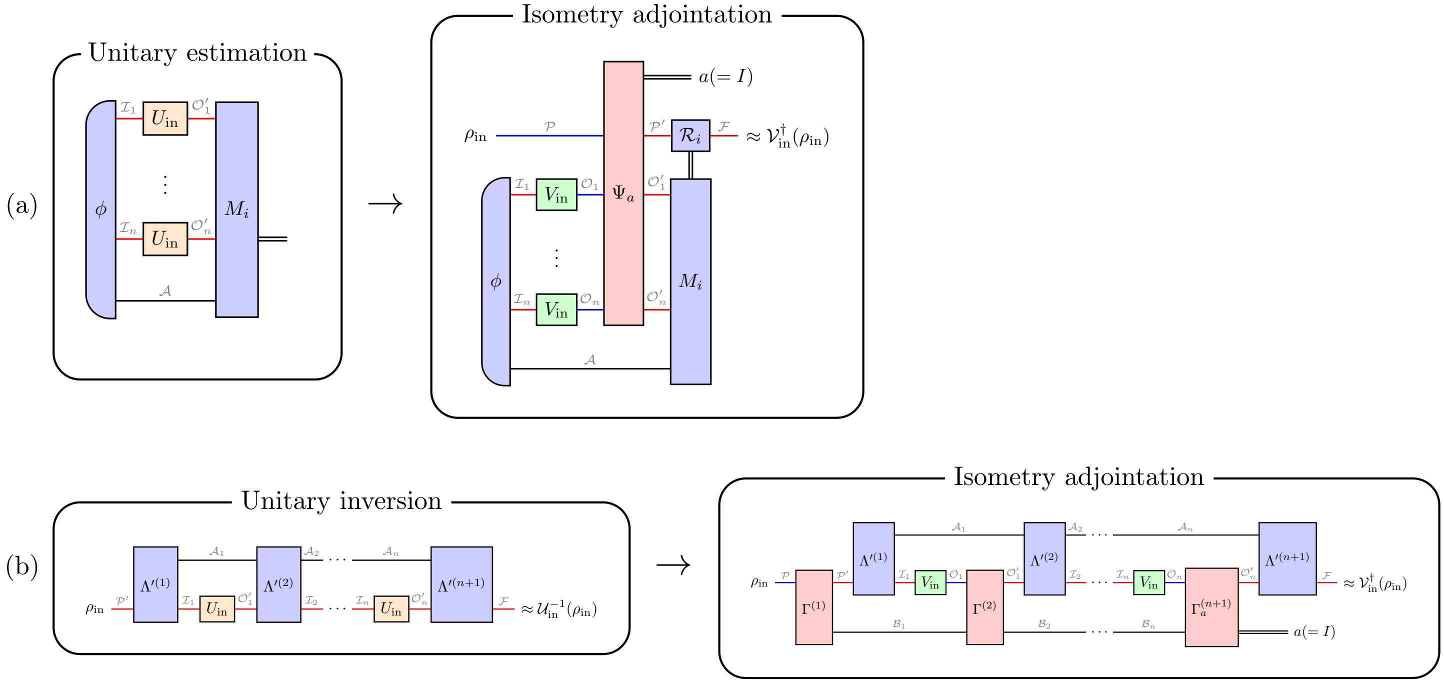

Figure 2. (a) Construction of a parallel isometry adjointation protocol from a covariant unitary estimation protocol. (b) Construction of a sequential isometry adjointation protocol from a unitary inversion protocol.

Our constructions of isometry adjointation protocols are transformations from the unitary inversion protocols. Since the unitary inversion protocols are transformations of quantum operations, or quantum supermaps [30], such transformations are called quantum supersupermaps. Using the idea of quantum supersupermaps, the problem to design an isometry adjointation protocol reduces to designing a unitary inversion protocol, which is extensively studied in previous works [15–19]. Note that a similar idea is used in Ref. [31], which presents transformation of the function applied on block-encoding unitary operation in quantum singular value transformations.

REDUCTION TO ISOMETRY INVERSION, UNIVERSAL ERROR DETECTION, AND PROGRAMMABLE PROJECTIVE MEASUREMENT

By discarding the measurement outcome of isometry adjointation protocols in Fig. 2, we can implement *isometry inversion* [32]. Isometry inversion is the task to implement the inverse operation $\mathcal{V}_{\mathrm{in}}^{-1}$ of the input isometry operation $V_{\mathrm{in}} \in \mathbb{V}_{\mathrm{iso}}(d, D)$, where the inverse operation is defined as a CPTP map such that $\mathcal{V}_{\mathrm{in}}^{-1} \circ \mathcal{V}_{\mathrm{in}} = \mathbb{1}_d$. We can show that the obtained isometry inversion protocol has the approximation error ϵ that is the same as the original unitary inversion protocol. Since d-dimensional unitary inversion with approximation error ϵ can be done using $n = O(\mathrm{poly}(d)\epsilon^{-1/2})$ (parallel) or $n = O(\mathrm{poly}(d) \log \epsilon^{-1})$ calls of the input unitary operation [17], we can construct the isometry inversion protocol with the same number of the input operations. Our construction with the deterministic exact unitary inversion for $d = 2$ [19] gives deterministic and exact isometry inversion.

By discarding the output state of isometry adjointation protocols in Fig. 2, we can implement *universal error detection*, which is a task to implement the POVM $\{\Pi_{\mathrm{Im}\, V_{\mathrm{in}}}, \mathbb{1}_D - \mathrm{Im}\, V_{\mathrm{in}}\}$ approximately. In particular, it implements the POVM $\{\Pi_{\mathrm{Im}\, V_{\mathrm{in}}} + \alpha(\mathbb{1}_D - \Pi_{\mathrm{Im}\, V_{\mathrm{in}}}), (1-\alpha)(\mathbb{1}_D - \Pi_{\mathrm{Im}\, V_{\mathrm{in}}})\}$, where α is given in (1), which quantifies the approximation error of the protocol. The minimal value of α among parallel protocols is explicitly given in Theorem 12 of the technical manuscript [1], which scales as $\inf_{\text{paralell protocol}} \alpha = \Theta(d^2/n)$. The special case ($d = 1$) of universal error detection reduces to a programmable projective measurement [20], which transforms an input unknown pure state $|\psi_{\mathrm{in}}\rangle \in \mathbb{C}^D$ to the corresponding POVM $\{|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|, \mathbb{1}_D - |\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|\}$. The optimal approximation error is obtained in Ref. [20], which corresponds to the $d = 1$ case of our explicit expression of α [25].

COMPARISON WITH THE PREVIOUS WORK [32] ON ISOMETRY INVERSION

We compare this work with a related previous work [32] on isometry inversion. Reference [32] proposes constructing an isometry inversion protocol from a parallel unitary inversion protocol. However, the construction presented in Ref. [32] does not work for a sequential unitary inversion protocol. In contrast, this work proposes a transformation from a sequential unitary inversion protocol to isometry inversion. This extension makes it possible to implement deterministic and exact isometry inversion using deterministic and exact unitary inversion [19, 33], which is known to be impossible by a parallel protocol [15–17]. Also, this work generalizes isometry inversion to isometry adjointation, which includes a meaningful task called programmable projective measurement [20] when we consider the special case $d = 1$.

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