Minimal Equational Theories for Quantum Circuits

– Non-Proceedings Submission –

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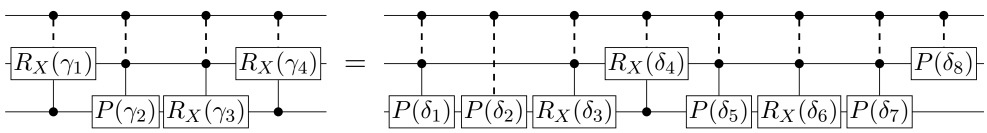
Abstract. We introduce the first minimal and complete equational theory for quantum circuits. Hence, we show that any true equation on quantum circuits can be derived from simple rules, all of them being standard except a novel but intuitive one which states that a multi-control 2π rotation is nothing but the identity. Our work improves on the recent complete equational theories for quantum circuits, by getting rid of several rules including a fairly impractical one. One of our main contributions is to prove the minimality of the equational theory, i.e. none of the rules can be derived from the other ones. More generally, we demonstrate that any complete equational theory on quantum circuits (when all gates are unitary) requires rules acting on an unbounded number of qubits. Finally, we also simplify the complete equational theories for quantum circuits with ancillary qubits and/or qubit discarding.

Since its introduction in the 1980s, the quantum circuit model has become ubiquitous in quantum computing, and a cornerstone of quantum software. Transforming quantum circuits is a central task in the development of the quantum computer, for circuit optimisation (such as minimizing the number of gates, T-gates, and CNot gates, as well as reducing circuit depth), for ensuring compatibility with hardware constraints, and for enabling fault-tolerant computations [15,19,20,21,22]. Such transformations can be achieved through the use of an equational theory, i.e. roughly speaking, a set of rules which allow one to replace a piece of circuit with an equivalent one. An equational theory is complete when any true equation can be derived. In other words, if two circuits represent the same unitary transformation, completeness ensures that they can be transformed into each other using only the rules of the equational theory.

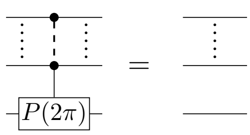
From a foundational perspective, a complete equational theory can be regarded as a set of axioms or principles that govern the behaviour of quantum circuits. Therefore, it is important to establish concise and meaningful rules that accurately capture the properties of quantum circuits.

Several equational theories have been demonstrated to be complete for specific, non-universal fragments of quantum circuits, like Clifford circuits [23], 1- and 2-qubit Clifford+T circuits [4], 3-qubit Clifford+CS [5], CNOT-dihedral circuits [1]. Recently, a first complete equational theory for arbitrary quantum circuits was introduced [9]. Derived from a complete equational theory for photonic circuits [8] through an elaborate completion procedure, this original complete equational theory for quantum circuits contained a few cumbersome rules, some of them have been shown to be derivable in [7]. There remained however a family of equations acting on an unbounded number of qubits, and involving a dozen parameters with non trivial relations among them (see Equation (E_{3D})), leaving open the question of whether such a family of intricate equations is necessary.



(E_{3D})

We introduce a concise and meaningful equational theory for quantum circuits, presented in its simplest form in Figure 1, where quantum circuits are considered up to global phases.[1] The main novelty, in terms of rules, is that the intricate rule (E_{3D}) is replaced by the following equation:



(I)

[1] When considering quantum circuits with global phases, the equational theory has two additional very simple equations that govern the behaviour of the global phases.

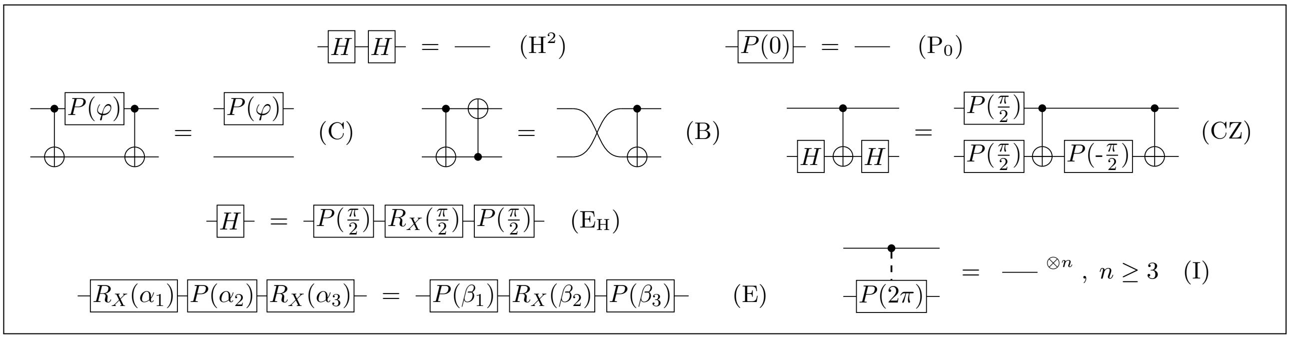


Fig. 1: Minimal and complete equational theory for quantum circuits up to global phases.

Semantically, (I) is trivial: a 2π Z-rotation is nothing but the identity, hence its controlled version is also the identity whatever the number of control qubits is. Syntactically, the multi-control gates are defined inductively and thus are shortcut notations for large circuits containing only basic generators $-\boxed{H}-$, $-\boxed{P(\varphi)}-$ and .

Each equation of Figure 1 has a simple and meaningful interpretation: (H^2) means that H is self inverse; (P_0) that a rotation of angle 0 is the identity; (C) that CNot is self inverse (when $\varphi = 0$), and also that $P(\varphi)$ and CNot commute on the control qubit; (B) essentially that composing 3 CNots is a swap; (CZ) that a Control-Z can be implemented in two ways using either one or two CNots; (E_H) is the Euler decomposition of H; and finally (E) relates two possible Euler decompositions into Z- and X-rotations.

Our main result is to prove that the equational theory we introduce in Figure 1 is complete and minimal. While completeness ensures that any valid equation involving quantum circuits can be derived from these rules, minimality guarantees that none of the equations in Figure 1 can be derived from the other equations.

Theorem 1. *The equational theory presented in Figure 1 is complete and minimal.*

In particular, for any $n_0 \geq 2$, the instance of (I) with n_0 control qubits cannot be derived from the other instances of (I) together with the other rules of Figure 1. Beyond the minimality of this particular equational theory, one of our main contributions is to show that there is no complete equational theories acting on a bounded number of qubits for vanilla[2] quantum circuits.

Theorem 2. *There is no complete equational theory for vanilla quantum circuits made of equations acting on a bounded number of qubits. More precisely any complete equational theory for n-qubit vanilla quantum circuits has at least one rule acting on n qubits.*

Minimality does not imply uniqueness. Depending on the context, it might be relevant to consider alternative equational theories. For instance, we show that one can replace the Equations (E_H) and (E) by

$$-\boxed{P(\varphi_1)}-\boxed{P(\varphi_2)}- \;=\; -\boxed{P(\varphi_1{+}\varphi_2)}- \qquad (P_+)$$

$$-\boxed{R_X(\alpha_1')}-\boxed{H}-\boxed{R_X(\alpha_3')}- \;=\; -\boxed{P(\beta_1')}-\boxed{R_X(\beta_2')}-\boxed{P(\beta_3')}- \qquad (E')$$

leading to an equational theory which is also complete and minimal. Equation (E') – introduced for the first time, up to our knowledge, in the context of the ZX-calculus [12,27] – is an alternative formulation of the Euler decomposition with only two parameters on the left-hand side of the equation.

We provide in this paper the first *minimal* equational theory for quantum circuits. Indeed, the question of the minimality is still open for the complete equational theories equipping non-universal fragments of quantum circuits, like Clifford [23], 2-qubit Clifford+T [4], and 3-qubit Clifford+CS [5]. More broadly, this is one of the first minimality results for a graphical language for quantum computing. Indeed, whereas the first completeness results for universal graphical quantum languages have been obtained through the ZX-calculus [16,17,13,18], and despite a great effort and significant progresses in the recent years, only *nearly minimal*[3] equational theories have been introduced for the

[2] By *vanilla* quantum circuits we mean that all gates are unitary, in particular there is no qubit initialisations, ancillary qubits or discarding.

[3] Here *nearly minimal* means that a majority of the rules, but not all, have been proved to be underivable form the other ones.

ZX-calculus [2,3,24,27] and its variants like the ZH-calculus [25,26]. In all these cases, the minimality of the provided complete equational theories is still open. Only the PBS-calculus is equipped with complete and minimal equational theories [10,11], notice however that the PBS-calculus focuses on coherent control and can only represent some specific oracle-based evolutions called superpositions of linear maps. In other words, the PBS-calculus can be seen as a construction to provide coherent control capabilities to arbitrary (graphical) quantum language.

Beyond vanilla quantum circuits, one can define equational theories for quantum circuits with ancilla and/or qubit discarding (or trace out). Various constructions exist to transport a complete equational theory for vanilla quantum circuits to these settings [14,6]. In [7], it has been shown that the intricate Equation (E_{3D}) can be derived from its 2-qubit case in the presence of ancillary qubits and/or discarding. We show that Equation (E_{3D}) is not necessary at all. Indeed we derive from the equational theory of Figure 1 simple complete equational theories, acting on a bounded number of qubits (namely at most 3), for quantum circuits with ancillary qubits (see Figure 2) and/or discarding (see Figure 3).

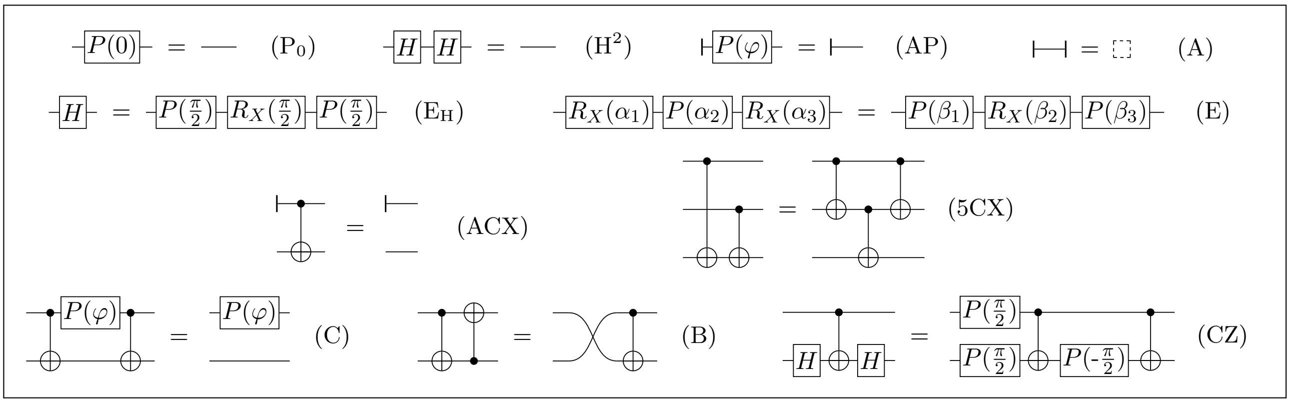


Fig. 2: Complete equational theory for quantum circuits with ancillae up to global phases.

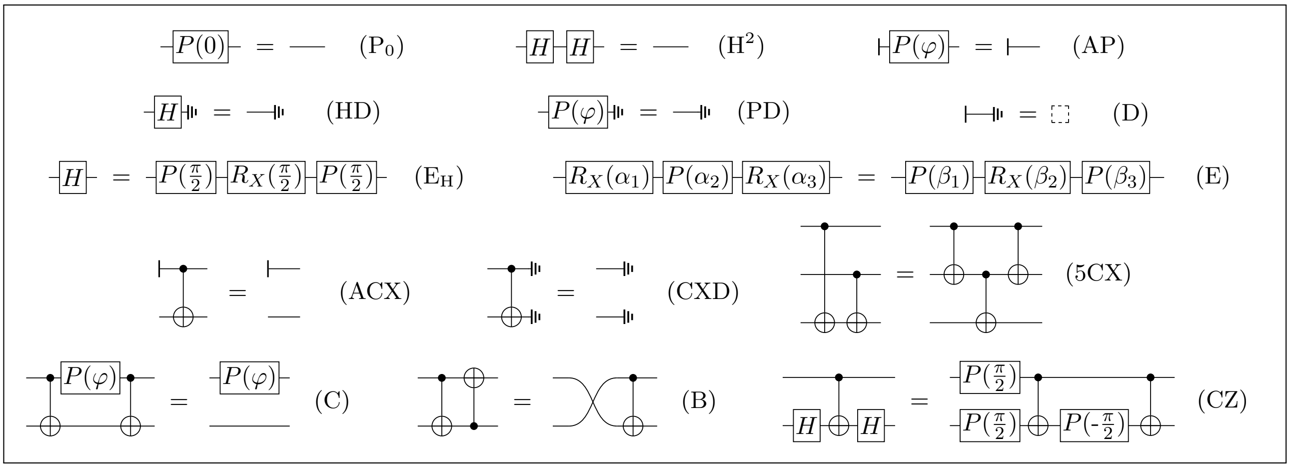


Fig. 3: Complete equational theory for quantum circuits with discard.

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