Characterising semi-Clifford gates using algebraic sets

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Abstract. Third-level gates are those that can be most easily implemented fault-tolerantly using magic states. Those that are semi-Clifford can be implemented with far more efficient use of resources. We prove that every third-level gate of up to two qudits is semi-Clifford. We thereby generalise results of Zeng-Chen-Chuang (2008) in the qubit case and of the second author (2020) in the qutrit case to the case of qudits of any prime dimension d.

Earlier results relied on exhaustive computations whereas our present work leverages tools of algebraic geometry which can be applied more widely within quantum information. Specifically, we construct two schemes corresponding to the sets of third-level Clifford hierarchy gates and third-level semi-Clifford gates. We then show that the two algebraic sets resulting from reducing these schemes modulo d share the same set of rational points.

1 Overview

Gates of the *third level of the Clifford hierarchy* play a central role in fault-tolerant quantum computation. Within the dominant stabiliser framework for quantum error correction, they are the natural choice for promoting the group of Clifford gates to a universal gate set. This is because they can be implemented fault-tolerantly using only Clifford gates and a preprepared magic state. The vast quantity of these resources states required for achieving fault-tolerance is a significant bottleneck for experimental implementations of quantum computers.

A subset of the third-level gates are *semi-Clifford*. These gates can be implemented using magic states of half the size needed for standard third-level gates. The urgency of reducing the overhead of fault-tolerance motivates the characterisation of semi-Clifford gates; of particular importance are the most common cases of one- or two-qubit gates. Research into the Clifford hierarchy and semi-Clifford gates has remained active from their discovery twenty-five years ago to the present [1, 3, 9, 12, 15, 16, 21, 22].

Zeng-Chen-Chuang [21] showed that all two-qubit third-level gates are semi-Clifford. The second author of the present work extended this result to qutrits [9]. In both cases, exhaustive computations that do not extend to higher dimensions were used.

The main result of the present work is:

Theorem 1 *For any odd prime dimension, every two-qudit third-level gate is semi-Clifford.*

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In order to establish our result for infinitely many dimensions simultaneously, our mathematical arguments are necessarily highly abstract. We employ tools of algebraic geometry—including some which have yet to be applied within quantum information.

We first transform our original question into one about two systems of polynomial equations over the finite field $\mathbb{Z}_d$. Therefore, our result follows from showing that two geometric spaces (specifically, *algebraic sets*), one arising from the set of third-level gates and the other from its subset of semi-Clifford gates, share the same set of (rational) points. Using schemes, an even more abstract notion of geometric space, we are able establish our result for all prime dimensions with only one series of computations.

The immediate impact of our results is to provide many more pathways towards efficiently achieving fault-tolerant universal quantum computing with qudits. Qudit-based computation promises increased capacity and efficiency [4, 19]. The advantages of qudit-based computation has led to rapidly accelerating development by experimentalists [5, 6, 13, 14, 17, 18, 20]. In the future, once quantum technology has progressed, we expect qudit-based computation to become commonplace.

Our work will also have a broader impact on quantum information as our mathematical methods are widely applicable to answering a range of structural questions within the stabiliser formalism.

The full preprint is available at `https://arxiv.org/abs/2309.15184` and is in press at *Communications in Mathematical Physics.*

2 Background

Building a large-scale practical quantum computer will require efficient methods of fault-tolerance and error correction. The most common family of quantum error correcting codes are stabiliser codes [11]. They are built up from Pauli gates and their eigenstates. Clifford gates, which preserve the Paulis under conjugation, are special in that they can be fault-tolerantly applied to encoded data. Quantum universality, however, further requires the ability to fault-tolerantly perform non-Clifford gates. Third-level gates can be deterministically and fault-tolerantly performed via the *gate teleportation* protocol using only Clifford gates supplemented with ancillary magic state resources [12]. Thus, non-Clifford third-level gates are critical to achieving fault-tolerant quantum computation.

A significant practical barrier to achieving quantum universality via the supplementation of Clifford gates with magic states is the need to prepare such states for every desired application of a non-Clifford gate. The original gate teleportation protocol implemented n-qubit third-level gates using magic states of $2n$ qubits. The need to reduce the burden of this substantial resource overhead cost led to the study of more efficient gate teleportation protocols.

Diagonal third-level gates can be implemented using magic states of only n qubits [22]. This was generalised to the 'nearly diagonal' semi-Clifford gates, i.e. those gates G such that $G = C_1 D C_2$ for C_1, C_2 being Clifford gates and D a diagonal gate [21]. It is known that all Clifford hierarchy gates of one or two qubits is semi-Clifford (via a proof involving exhaustive computation) and that, for more qubits, there are gates in every level that are not semi-Clifford [3, 21].

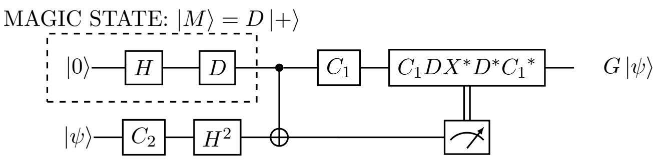

Figure 1: Efficient gate teleportation protocol for a one-**qudit** semi-Clifford gate $G = C_1 D C_2$.

Efficient gate teleportation was first studied in the qudit setting by the second author of the present work [9]. In this work, an abstract formalism for studying the Clifford hierarchy based on the foundational Stone-von Neumann theorem was introduced. This framework and several of the results derived from it play an essential role below. It was shown that all two-qutrit third-level gates are semi-Clifford. However, this proof also required exhaustive computation due to the mathematical difficulty of establishing it analytically.

3 Tools and Methodology

3.1 Quantum information

Conjugate tuples Let Z_i, X_i resp. be the Pauli gate Z, X resp. on the i-th qudit and identity on all others. Then any gate G yields an ordered set of $2n$ gates $\{(GZ_iG^*, GX_iG^*)\}_{i\in[n]}$ by conjugating the basic Paulis. Since Z_i, X_i obey the Weyl commutation relations, so do their conjugated versions. Remarkably, this process can be reversed: from any ordered set of $2n$ gates obeying the Weyl commutation relations, the gate G can be reconstructed up to phase. It is shown in [9], extending Beigi-Shor [3], that gates of the Clifford hierarchy are most fruitfully studied via their conjugate tuple.

Simplified third-level gates We call a third-level gate *simplified* if its conjugate tuple takes a simple form: every member is the product of a diagonal Clifford gate and a Pauli gate. From Cui-Gottesman-Krishna [8], we know that diagonal Clifford gates correspond to quadratic polynomials over the finite field $\mathbb{Z}_d$. By applying a classification of maximal abelian subgroups of the symplectic group due to Barry [2], we can show that every third-level gate can be conjugated by a Clifford gate to give a simplified one. We can thereby work with discrete combinatorial data to describe a third-level gate rather than a complex unitary matrix. We characterise which tuples of elements of $\mathbb{Z}_d$ arise from simplified third-level gates using a family of polynomial equations $\mathcal{F}_1$ and those which are semi-Clifford with an additional such family $\mathcal{F}_2$.

3.2 Algebraic geometry

Nullstellensatz The interplay between algebra and geometry that forms the basis of algebraic geometry arises from a correspondence between the solution sets of polynomial equations and certain kinds of ideals in a ring of polynomials called radical ideals. Having transformed our original problem concerning third-level and semi-Clifford gates into one concerning the equivalence of two geometric spaces, we use the Nullstellensatz to transform the problem into one concerning the equality of two radical ideals.

Computational algebra The advantage of the algebraic formulation of our problem is that it can be solved by a series of computations. In particular, we establish the required equality between the solution space of $\mathcal{F}_1$ and $\mathcal{F}_1 \cup \mathcal{F}_2$ by analysing a decomposition of their corresponding ideals. This is enabled using techniques involving *Groebner bases* [7] that can be performed using the computational algebra system `Magma`. Applying these techniques are far from straightforward, however, as they are feasible only for systems far simpler than the ones we are interested in.

Schemes Introduced by Grothendieck in 1960, schemes [10] are geometric spaces that generalise algebraic sets in order to provide a foundation for modern algebraic geometry and number theory. They are analogous to differentiable manifolds; instead of looking locally like a space on which one can use calculus (e.g. $\mathbb{R}^n$), a scheme looks locally like a space on which one can use algebra (in a general commutative ring). The language of schemes motivates and provides a framework to establish our result for infinitely many dimensions at once.

4 Main Results

We summarise our main results at a high level as a series of steps towards establishing Theorem 1.

1. We show that all two-qudit third-level gates are semi-Clifford if and only if all *simplified* two-qudit third-level gates are semi-Clifford.
2. We show that all simplified third-level gates yield a tuple of elements of $\mathbb{Z}_d$ that is a solution to a family $\mathcal{F}_1$ of polynomial equations.
3. We show that a simplified third-level gate is semi-Clifford if its tuple of elements of $\mathbb{Z}_d$ also satisfies a second family $\mathcal{F}_2$ of polynomial equations.
4. Using the Nullstellensatz, the our main theorem will follow from showing that the two radical ideals that correspond to the two algebraic sets constructed from third-level and semi-Clifford gates are the same. In principle, the computation of radical ideals can be performed algorithmically, e.g. by a computer algebra system such as `Magma`.
5. In practice, the families $\mathcal{F}_1, \mathcal{F}_1 \cup \mathcal{F}_2$ of polynomial equations are far too complex and involve far too many variables to be amenable to computation. We therefore perform a series of reductions that decrease the number of variables involved and replace our families of polynomial equations with simpler ones. For example, we weaken a family of quadratic equations and replace them with a linear system. We show that consistency of this system requires satisfying one of two highly complex polynomials in fewer variables.
6. Simplifying our systems of equations comes at the cost of requiring additional mathematical arguments to show that our feasible computation establishes that two-qudit third-level gates are semi-Clifford, which we supply.
7. Technically, one would have to perform a different computation for each odd prime dimension d. We provide an argument, using the language of schemes, that allows a single computation to establish our result for all dimensions simultaneously. Specifically, we demonstrate the isomorphism of two schemes that correspond to third-level and semi-Clifford gates; our result follows by reducing these schemes to deduce the desired equality of algebraic sets for each dimension d.

5 Impact and Future Work

A natural follow-up question to our work is to generalise our result to higher levels of the Clifford hierarchy. We can also extend our techniques to generalise counterexamples of Zeng-Chen-Chuang [21] and Gottesman–Mochon to higher dimensions; that is, find examples of n-qudit k-th level gates that are not semi-Clifford when $n > 2, k > 3$ or $n > 3, k = 3$.

This work significantly advances the program of classifying gates of the Clifford hierarchy and semi-Clifford gates. Deeper mathematical understanding of the Clifford hierarchy and semi-Clifford gates will lead to more efficient circuit and gate synthesis. It further bolsters the viability of qudit-based fault-tolerant universal quantum computers by providing complete sets of efficient gate teleportation protocols. This is practically important as qudit magic state distillation has been proposed as a significantly more efficient alternative to the qubit case [4]. This, and other advantages of qudits, are driving current experimental research.

The abstract mathematical techniques developed to solve our problem are widely applicable to many more problems within quantum information. We give a blueprint for solving any problem that can be recast in terms of the equivalence of solution sets of polynomial equations over $\mathbb{Z}_d$. This is a potentially very broad class of problems given that the dominant stabiliser formalism for quantum error correction is based on the standard representation of the Heisenberg-Weyl group over $\mathbb{Z}_d$.

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