Tsirelson bounds for quantum correlations with indefinite causal order

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1 Extended abstract

Quantum theory is in principle compatible with processes that violate causal inequalities, an analogue of Bell inequalities that constrain the correlations observed by parties operating in a definite order [1–3]. To date, many examples of causal inequalities that are potentially violated by processes with indefinite causal order have been found [2, 4–11]. However, in general, the maximum quantum violations of these inequalities are still unknown even in the simplest cases, unlike in the case of Bell inequalities such as Clauser-Horne-Shimony-Holt (CHSH) inequality [12], for which Tsirelson bound provides the ultimate violation achievable in quantum theory. The lack of exact bounds on the quantum violation of causal inequalities limits our understanding of indefinite causal order in quantum mechanics. In addition, new questions have recently arisen from the introduction of a new class of scenarios where not only the causal order of the experiments, but also the temporal direction of the information flow within the local laboratories can be indefinite [13]. Can these scenarios lead to even larger violations? And in the affirmative case, where does the boundary lie between the correlations achievable with indefinite causal order alone and those achievable when indefinite causal order is combined with indefinite temporal direction?

Here we answer all the above questions. First, we develop a general method for bounding the violation of causal inequalities by quantum processes with indefinite causal order. We start by showing that the maximal violation of a special class of causal inequalities, termed *single-trigger causal inequalities*, can be determined explicitly. The maximal violation of single-trigger causal inequalities provides upper bounds on the violations of arbitrary causal inequalities. Mathematically, these upper bounds can be seen as an semidefinite programming (SDP) relaxation of the original problem of computing the maximal quantum violation of causal inequalities.

Using this method, we establish the analogue of Tsirelson bound for paradigmatic examples of causal inequalities. We show the maximal violation of the Oreshkov-Costa-Brukner (OCB) inequality [2] and the inequality associated with Lazy Guess Your Neighbor's Input game [5]. In addition, we provide a non-trivial upper bound of the success probability of Guess Your Neighbor's Input game [5]. Our results allow for a geometric representation of the quantum correlations arising from indefinite causal order. Intriguingly, we find that the geometric representation of the OCB correlations coincides with the representation of the CHSH correlations in the Bell inequality setting [14].

Then, we show that classical processes with indefinite causal order and time direction can violate all causal inequalities to their algebraic maximum. These processes are the classical version of the quantum processes with indefinite time direction introduced in Ref. [13]. They are in principle compatible with the validity of classical physics in the laboratories of the different parties, but do not assume a privileged direction of time outside each laboratory. In particular, we construct a classical process which allows two parties to perfectly signal to each other.

Our results offer new insights into the structure of the set of quantum correlations generated by quantum indefinite-causal-order processes, and can be used as a tool to better understand the operational implication of indefinite causal order in quantum

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theory. An open question is whether our general bound could be tight for all the other causal inequalities. The analogy with Bell inequalities, however, suggests a negative answer. In Bell scenarios, a converging sequence of upper bounds on the value of maximal quantum violations is provided by the Navascués-Pironio-Acín SDP hierarchy [15, 16]. The analogy with this situation suggests that our SDP relaxation may be just the first level of a a similar hierarchy of SDPs. Determining whether this analogy is correct, and, in the affirmative case, identifying the other levels of the hierarchy are among the most important research directions opened by our work. Another interesting direction is to extend our method for the calculation of the ICO bound to other type of inequalities with non-trivial causal structure, such as the inequalities recently studied in Refs. [17, 18]. Another interesting direction of future research is to establish self-testing results for causal inequalities, in analogy to the self-testing results in Bell scenarios [19, 20]. Such a self-testing result may have cryptographic implications, in a similar way as it was observed in the setting of Bell correlations. While the physical realization of the OCB process is still an open problem, these implications would provide important foundational insights into the operational understanding of indefinite causal order in quantum theory. Finally, our results open up a search for physical principles capable of explaining why the violation of causal inequalities by ICO quantum processes is not equal, in general, to the algebraic maximum, and, of determining the exact value of the quantum violation. In the context of Bell inequalities, the analogue question was originally raised by Popescu and Rohrlich [21], and led to the discovery of new information theoretic principles, such as non-trivial communication complexity [22–24], non-trivial nonlocal computation [25], information causality [26], macroscopic locality [27], and local orthogonality [28].

2 Technical version of the work

Our paper is available on arXiv (`https://arxiv.org/abs/2403.02749`).

References

[1] G. Chiribella, G. D'Ariano, P. Perinotti, and B. Valiron, "Beyond quantum computers," *arXiv preprint arXiv:0912.0195*, 2009.

[2] O. Oreshkov, F. Costa, and Č. Brukner, "Quantum correlations with no causal order," *Nature communications*, vol. 3, no. 1, pp. 1–8, 2012.

[3] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, "Quantum computations without definite causal structure," *Physical Review A*, vol. 88, no. 2, p. 022318, 2013.

[4] O. Oreshkov and C. Giarmatzi, "Causal and causally separable processes," *New Journal of Physics*, vol. 18, no. 9, p. 093020, 2016.

[5] C. Branciard, M. Araújo, A. Feix, F. Costa, and Č. Brukner, "The simplest causal inequalities and their violation," *New Journal of Physics*, vol. 18, no. 1, p. 013008, 2015.

[6] Ä. Baumeler, A. Feix, and S. Wolf, "Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios," *Physical Review A*, vol. 90, no. 4, p. 042106, 2014.

[7] Ä. Baumeler and S. Wolf, "Perfect signaling among three parties violating predefined causal order," in *2014 IEEE International Symposium on Information Theory*, pp. 526–530, IEEE, 2014.

[8] Ä. Baumeler and S. Wolf, "The space of logically consistent classical processes without causal order," *New Journal of Physics*, vol. 18, no. 1, p. 013036, 2016.

[9] S. S. Bhattacharya and M. Banik, "Biased non-causal game," *arXiv preprint arXiv:1509.02721*, 2015.

[10] A. Feix, M. Araújo, and Č. Brukner, "Causally nonseparable processes admitting a causal model," *New Journal of Physics*, vol. 18, no. 8, p. 083040, 2016.

[11] A. A. Abbott, C. Giarmatzi, F. Costa, and C. Branciard, "Multipartite causal correlations: polytopes and inequalities," *Physical Review A*, vol. 94, no. 3, p. 032131, 2016.

[12] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, "Proposed experiment to test local hidden-variable theories," *Physical review letters*, vol. 23, no. 15, p. 880, 1969.

[13] G. Chiribella and Z. Liu, "Quantum operations with indefinite time direction," *Communications Physics*, vol. 5, no. 1, pp. 1–8, 2022.

[14] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, "Bell nonlocality," *Reviews of modern physics*, vol. 86, no. 2, p. 419, 2014.

[15] M. Navascués, S. Pironio, and A. Acín, "Bounding the set of quantum correlations," *Physical Review Letters*, vol. 98, no. 1, p. 010401, 2007.

[16] M. Navascués, S. Pironio, and A. Acín, "A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations," *New Journal of Physics*, vol. 10, no. 7, p. 073013, 2008.

[17] S. Gogioso and N. Pinzani, "The geometry of causality," *arXiv preprint arXiv:2303.09017*, 2023.

[18] T. van der Lugt, J. Barrett, and G. Chiribella, "Device-independent certification of indefinite causal order in the quantum switch," *Nature Communications*, vol. 14, no. 1, p. 5811, 2023.

[19] D. Mayers and A. Yao, "Quantum cryptography with imperfect apparatus," in *Proceedings 39th Annual Symposium on Foundations of Computer Science (Cat. No. 98CB36280)*, pp. 503–509, IEEE, 1998.

[20] D. Mayers and A. Yao, "Self testing quantum apparatus," *Quantum Info. Comput.*, vol. 4, p. 273–286, jul 2004.

[21] S. Popescu and D. Rohrlich, "Quantum nonlocality as an axiom," *Foundations of Physics*, vol. 24, pp. 379–385, 1994.

[22] W. Van Dam, *Nonlocality and communication complexity.* PhD thesis, University of Oxford, 1999.

[23] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, "Limit on nonlocality in any world in which communication complexity is not trivial," *Physical Review Letters*, vol. 96, no. 25, p. 250401, 2006.

[24] N. Brunner and P. Skrzypczyk, "Nonlocality distillation and postquantum theories with trivial communication complexity," *Physical review letters*, vol. 102, no. 16, p. 160403, 2009.

[25] N. Linden, S. Popescu, A. J. Short, and A. Winter, "Quantum nonlocality and beyond: limits from nonlocal computation," *Physical review letters*, vol. 99, no. 18, p. 180502, 2007.

[26] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, "Information causality as a physical principle," *Nature*, vol. 461, no. 7267, pp. 1101–1104, 2009.

[27] M. Navascués and H. Wunderlich, "A glance beyond the quantum model," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 466, no. 2115, pp. 881–890, 2010.

[28] T. Fritz, A. B. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín, "Local orthogonality as a multipartite principle for quantum correlations," *Nature communications*, vol. 4, no. 1, p. 2263, 2013.