A minimalist approach to operational theories

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This abstract refers to the results reported in Ref. [1] (arXiv link) and Ref. [2], which is in preparation.

We explore the concept of minimalism within the framework of Operational Probabilistic Theories (OPTs) [3–6]. We show that restricting to the bone the set of allowed operations, even when allowing for conditioning if entanglement is present, implies that generic theories of information processing satisfy the property of No-Information Without Disturbance (NIWD) [4, 7, 8]—the fact that extracting information from a system irreversibly alters its state—and do not admit a broadcasting channel [2, 9–11]. Furthermore, we show that this holds true even when classicality is assumed by explicitly building two theories, termed Minimal Classical Theory (MCT) [1] and Minimal Strongly-causal Bilocal Classical Theory (MSBCT) [2].

Our work is part of the literature that aims at understanding which properties and/or advantages can be possessed by operational theories that do not satisfy common assumptions made in the frameworks of operational theories [3–6, 12–16]. For example, what happens when restrictions are posed on the measurements that can be implemented in a theory is studied in Refs. [17, 18]. While, Ref. [19] introduces the class of non-free operational theories that are not closed under composition and Ref. [20] discusses accessible GPT fragments, whose aim is to study scenarios where the states and the effects of a theory are restricted to just those accessible by a fixed experimental scenario.

Working within the framework of OPTs, we study the properties of a particular class of operational theories, called Minimal Operational Probabilistic Theories (MOPTs) [1], in which strong restrictions are placed on the implementable transformations with nontrivial input and output systems. More in detail, the only allowed transformations are those that can be obtained as parallel and sequential composition of preparations, measurements, the identity and the swap. We remark that in MOPTs there are no requirements made on the spaces of states and effects. Our results hold regardless of how they are defined, so they could also satisfy restrictions such as those introduced in Refs. [17, 18].

We demonstrate that the identity transformation in every causal MOPT is atomic—meaning that it can be obtain only as coarse graining of transformations that are proportional to the identity itself—, which implies that these theories satisfy NIWD [1, 8], have irreversibility of measurement disturbance [1] and do not admit a broadcasting channel [2].

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Furthermore, we introduce MCT, which is a classical MOPT. More in detail, MCT is a minimal version of Classical Theory (CT), meaning that is a MOPT that has the same states and effects spaces as CT.

In our works, classical theories are defined as simplicial theories (i.e., theories where the states spaces are simplexes) whose pure states are jointly perfectly discriminable [6, 21]. This also makes MCT both Kochen-Specker noncontextual [22, 23] and generalised noncontextual [14, 16, 24].

Given that it is a minimal theory, MCT satisfies all the properties just listed. Therefore, MCT proves that NIWD, irreversibility and not admitting a broadcasting channel cannot be understood *per se* as signatures of non-classicality. Furthermore, this theory serves as an example where irreversibility of measurement disturbance holds, even though all effects are compatible i.e. jointly implementable. [1, 25].

We highlight that our result differs from the one proposed in Ref. [26]. In that reference, the authors introduce a theory with operations causing irreversible disturbance to the systems they act upon. However, this theory only exhibits some effects that are compatible with all others. Therefore, it fails to establish the complete independence of the presence of incompatible measurements from irreversibility, as it does not satisfy full compatibility of all observation instruments.

Although MCT serves as a powerful example to prove the independence between some physical properties, it is nonetheless a bit unsatisfactory. This is because it is a theory in which it is not even possible to condition which experiment to perform based on the result of a previous experiment—in other words it does not satisfy the property of strong causality, also referred to as classical control on outcomes and post-processing [4–6, 27]—, which is a very natural property to be expected of any theory that attempts to model our physical world.

Therefore, we constructed MSBCT and the more general class of OPTs it belong to; the class of Minimal Strongly-causal Operational Probabilistic Theories (MSOPTs) [2], which are MOPTs to which all conditioned operations are added.

MSOPTs, given how they have been defined, allow to characterise what properties of operational theories derive solely from satisfying the property of strong causality.

In our work we prove that whenever this class of theories admit entangled states that cannot be perfectly discriminated from all separable states, the identity transformation is atomic. Therefore, also in this case they satisfy NIWD, irreversibility and do not admit a broadcasting channel.

While it is not surprising that the property of conditioning alone could not guarantee the violation of NIWD or the presence of a broadcasting channel—Quantum Theory (QT) admits all conditioned operations, satisfies NIWD and does not admit a broadcasting channel—, it is a bit unexpected that this still holds when assuming classicality. However, MSBCT is an example of this. In fact, this theory is both classical and a MSOPT. Thus, even under the strong causality hypothesis NIWD, irreversibility and not admitting a broadcasting channel cannot be understood *per se* as signatures of non-classicality.

In more detail, MSBCT is a minimal version of Bilocal Classical Theory (BCT) [6] to which all the possible conditioned operations are added. Other than being classical, MSBCT is also Kochen-Specker noncontextual. However, in this case it is not yet known whether or not it is also generalised noncontextual.

BCT is a bilocal version of CT, meaning that to fully characterise the state of a composite system AB one also has to perform measurements on the whole composite system, since the local measurements on only the subsystems A and B are not sufficient. More generally, this property is called *bilocal discriminability* [6, 21], while the possibility of completely characterizing the states of a theory performing only local measurements is referred to as *local discriminability* [4, 6, 21, 28]. Example of another theory satisfying the former property is Fermionic Quantum Theory [29], while QT, CT and MCT all satisfy local discriminability.

The reason why a strongly causal version of MCT could not be used as counterexample also in this case is that by adding all conditioned operations to MCT one recovers CT. This is due to the fact that the conditioned operations allow to recover all the reversible transformations of CT and, consequently, all the transformations of the latter theory through their reversible dilatation.
This also proves that the three properties of simpliciality, strong causality and local discriminability uniquely determine CT amongst all other OPTs. Furthermore, this set of properties represents a minimal set of axioms, since, due to the fact that it is possible to construct theories that satisfy just two of the three, these three properties are logically independent. QT is an example of a theory that is non-classical while being strongly causal and satisfying local discriminability. MCT is a non-strongly causal simplicial theory satisfying local discriminability. And, last but not least, MSBCT is strongly casual and simplicial, but it does not satisfy local discriminability.

We conclude by making a few remarks.

The fact that MCT and MSBCT do not admit a broadcasting channel even though they are simplicial is not in contrast with the results of Refs. [9, 10], since these results are obtained under the hypothesis of local discriminability, which excludes MSBCT, and that some conditioned operations are allowed, which excludes MCT.

Furthermore, we observe that the class of MOPTs allows to construct theories by only worrying about states and effects spaces. This is similar to what is done in the prepare-and-measure scenarios often considered in the Generalized Probabilistic Theories (GPTs) literature [16, 30]. Therefore, given that starting from any GPT defined in the prepare-and-measure scenario it is always possible to construct a MOPT, MOPTs represent a possible bridge bridge between the frameworks of OPTs and GPTs.

Last but not least, we observe that further studies on MOPTs and MSOPTs could lead to new discoveries about the cryptographic and computational properties of generic operational theories. With respect to the former topic, the idea is that these classes of theories, given that they satisfy the NIWD property, could allow to study which properties guarantee the existence of information-theoretically secure key distribution cryptographic protocols. The strong restrictions on the set of transformations would allow for greater control over all the variables involved in a communication protocol. For the latter, MSOPTs open the possibility of studying the paradigm of measurement-based computation in an operational setting, given that this computational paradigm consists in carrying out a series of conditioned measurements on an appropriate initial entangled state.

Summing up our results, in our work we show that even when classicality is assumed it is possible to formulate theories in which an interaction aimed at obtaining information about a physical system causes uncontrollable and irreversible disturbance in the system being observed, contrary to what was believed even by Heisenberg [31]. Furthermore, we also show that even the possibility of broadcasting is not always guaranteed. Therefore, NIWD and the absence of a broadcasting channel cannot be understood *per se* as signatures of non-classicality.

The classes of MOPTs and MSOPTs are also introduced. These theories, on the one hand, could represent a bridge between OPTs and GPTs based on the measure-and-prepare scenario. On the other hand, their study could increase our understanding of the properties that allow for a generic physical theory to admit information-theoretically secure cryptographic protocols for key distribution and to further characterise the computational properties of generic operational theories.

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