Characterizing Signalling: Connections between Causal Inference and Space-time Geometry

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Causality is pivotal to our understanding of the world, presenting itself in various forms: information-theoretic and relativistic, the former linked to the flow of information, and the latter to the structure of spacetime. Leveraging a framework introduced in PRA, 106, 032204 (2022), which formally connects these two notions in general physical theories, we study their interplay. Here, information-theoretic causality is defined through a causal modelling approach, which enables the inference of causal connections through agents' interventions and correlations. First, we improve the characterization of information-theoretic signalling as defined through so-called affects relations. Specifically, we provide conditions for identifying redundancies in different parts of such a relation, introducing techniques for causal inference in unfaithful causal models (where the observable data does not "faithfully" reflect the causal dependences). In particular, this demonstrates the possibility of causal inference using the absence of signalling between certain nodes. Second, we define an order-theoretic property called conicality, showing that it is satisfied for light cones in Minkowski space-times with $\mathrm{d} > 1$ spatial dimensions but violated for $\mathrm{d} = 1$. Finally, we study the embedding of information-theoretic causal models in spacetime without violating relativistic principles such as no superluminal signalling (NSS). In general, we observe that constraints imposed by NSS in a spacetime and those imposed by purely information-theoretic causal inference behave differently. We then prove a correspondence between conical space-times and faithful causal models: in both cases, there emerges a parallel between these two types of constraints. This indicates a connection between informational and geometric notions of causality, and offers new insights and tools for studying the relations between the principles of NSS and no causal loops in different spacetime geometries and theories of information processing.

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1 Introduction

Scientific enquiry hinges on causally explaining our observations. Causal modeling and causal inference, originating in classical statistics [1, 2], provide rigorous mathematical frameworks for doing so, and its versatility has led to applications across classical data driven disciplines [2, 3, 4, 5, 6, 7]. The approach has also seen generalizations to quantum theory, enabling successful causal explanations of fundamental correlations and phenomena in the quantum world [8, 9]. Causal models enable an operational definition of causality in terms of information flow, discernible through agents' interventions, without reference to space or time. However, in relativistic physics, causality is intrinsically linked to the geometry of space and time. These two notions must be compatible, as space-time structures constrain information flow through principles of relativistic causality, such as the impossibility of superluminal signaling (NSS). However, this interplay is not fully understood.

Recognizing the imperative to bridge this gap, recent work [10] develops a causal modelling formalism (which we will refer to as the *affects framework*) applicable to a general class of physical theories, and relates it to space-time structure to formalize relativistic principles. The concept of *(higher-order) affects relations* was introduced to capture the general possibilities for agents to signal to each other, through interventions and correlations. The usual modelling of signalling in specific quantum information protocols like the Bell scenario, only refer to correlations. However, to fully capture signalling in general scenarios, correlations do not suffice and one must also account for interventions, which the affects framework does [11]. The formalism encompasses scenarios with cyclic and fine-tuned causal influences and latent classical/non-classical causes. Here, fine-tuning refers to the possibility of carefully tuned causal mechanisms that wash out certain correlations and signalling possibilities, making the observed data not "faithful" to the underlying causal dependencies. As a consequence, causation does not imply signalling in the presence of fine-tuning, making causation, signalling and correlations three inequivalent concepts in general. Fine-tuning is crucial in practical scenarios, such as in the security of cryptographic protocols, however there are few techniques for accounting for fine-tuning in the context of causal inference, and for clearly distinguishing the inequivalences introduced by fine-tuning.

Further, by embedding a causal model in space-time, the principle of no superluminal signalling (NSS) is formalized through the compatibility between the higher-order affects relations and the light-cone structure of the space-time. The affects framework led to the surprising finding that causal loops can be embedded in $1+1$-Minkowski space-time without leading to superluminal signalling, even when the existence of the loop can be operationally verified through interventions [12]. Whether such loops are possible in higher-dimensional Minkowski space-time, as well as the necessary and sufficient conditions for ruling out such causal loops in a physical theory remain important open questions. More generally, the affects framework provides a platform to study what kind of properties are common to informational and spatio-temporal notions of causality in different theories, which is relatively little explored, although relevant for fundamental and practical considerations alike.

Addressing such questions requires a careful characterization of (a) the operational properties of higher-order affects relations and (b) order-theoretic properties relating to space-time geometry, and their interplay. This also calls for effective tools for handling fine-tuning, as several relativistic principles and information-theoretic notions become inequivalent when allowing for fine-tuning [10, 12, 13].

2 Summary of contributions

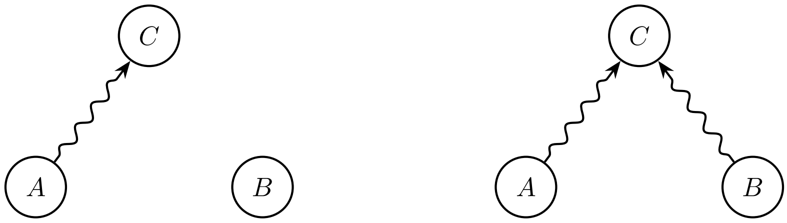
We summarize our main contributions, after briefly describing affects relations. A (higher-order) affects relation, denoted as $X \vDash Y \mid \mathrm{do}(Z)$, carries three arguments, which are disjoint sets of random variables (RVs) and it captures that an agent who intervenes on X can signal to an agent who can observe data on Y and is given information about interventions performed on Z. These random variables represent classical settings and outcomes that may be obtained by operating on fundamentally non-classical systems of a general theory. Depending on whether Z is empty or not, we have a zeroth-order or a higher-order relation, and we will simply write $X \vDash Y$ in the former case. Here, X and Z are interventional arguments while Y is a purely observational argument.

1. Causal inference and fine-tuning: Generally, such an affects relation can carry some redundancies which limit causal inference. To motivate this, consider the simple causal structure given in Figure 1a and a classical causal model where A and B are distributed according to some priors $P(A)$ and $P(B)$ while the dependence of C on A is $C = A$. Here, we would have $A \vDash C$ and $AB \vDash C$, but we would expect the latter relation to be "reducible" to the former, since B is completely redundant here and adds no information. Generally, one needs a criterion for identifying such redundancies without knowing the causal structure but only given observed data such as correlations and observable affects relations.

We provide such criteria by formally defining *reducibility* for every argument of an affects relation[1], that identifies operationally whether the relation can be reduced to an informationally equivalent relation on sets of smaller cardinality. We then show that for any affects relation $X \vDash Y \mid \mathrm{do}(Z)$ that is irreducible in its first and third argument, each RV $e_{XZ} \in X \cup Z$ will be a cause of Y, while this is not true for reducible relations (as the example of Figure 1a illustrates).

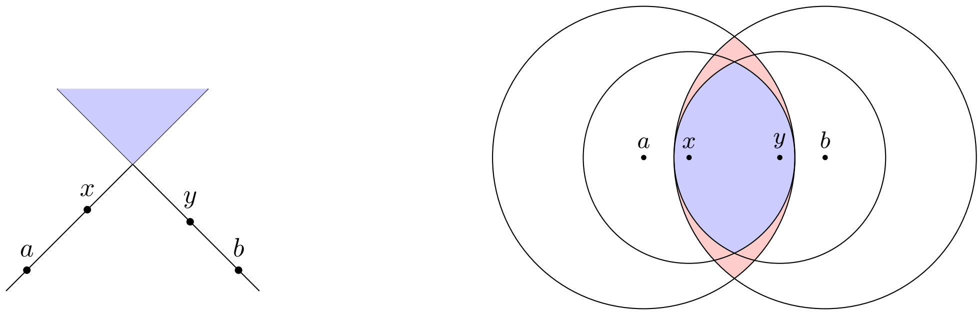
Next, we introduce the notion of *clustering*, which corresponds to the existence of an affects relation $X \vDash Y \mid \mathrm{do}(Z)$, without the existence of relations of the same form involving strict subsets of X, Y or Z. As an example, consider the causal structure of Figure 1b and a classical causal model where all nodes are binary variables, A and B are distributed uniformly and C is obtained by taking the XOR of its two causes, $C = A \oplus B$. Here, C carries no information about A or B individually but only about their joint correlations. Specifically, we would have $AB \vDash C$ but $A \not\vDash C$ and $B \not\vDash C$ i.e., the relation $AB \vDash C$ is clustered in the first argument, since no subset of this argument affects C. Moreover, notice that we

[1]Reducibility of affects relations in the first interventional argument was introduced in [10].



(a) (b)

Figure 1: Examples of information-theoretic causal structures: circled nodes correspond to observed random variables, and edges $\rightsquigarrow$ denote causal influences. More generally we can also have uncircled nodes, which correspond to latent (possibly non-classical causes). The text illustrates the novel concepts of reducibility and clustering through examples of classical causal models on these causal structures.



(a) d=1 (b) d=2

Figure 2: Schematic representation of light cones in Minkowski space-time for d spatial dimensions. (a) For $d = 1$ Minkowski space-time does not show conicality. This is due to the joint futures of the $L_1 := \{a, b\}$ and $L_2 := \{x, y\}$ being identical: $f(L_1) = f(L_2)$. (b) For $d = 2$, Minkowski space-time shows conicality. As can be seen from the figure which represents one particular time slice, the joint futures are distinct, as $f(L_1) \neq f(L_2)$.

expect $AB \vDash C$ in this case to be irreducible since both A and B equally contribute to determining C, neither are redundant (in contrast to the example in Figure 1a). We generally define clustering for any scenario and every argument of an affects relation, showing that clustering implies irreducibility and is an operational witness of the fine-tuning of the underlying causal mechanisms.

Using this, we differentiate between at least three types of operationally detectable fine-tunings, and also derive applications of these new concepts for causal inference. While typical causal inference results and algorithms assume no fine-tuning due to complications that arise otherwise [2, 14], our results in particular show that in certain fine-tuned models, the *absence* of signalling between some of the nodes can be employed for inferring additional causal connections.

2. Order-theoretic properties of space-time: We introduce an order-theoretic property of the causal structure of space-time (modelled as a partially ordered set), which is called *conicality*. We show that $d+1$-dimensional Minkowski space-times with $d > 1$ are conical space-times, while this is not the case for 1+1-dimensional Minkowski space-time (see Figure 2). Conicality captures the requirement that for a set L of space-time points, their joint future $f(L)$ (intersection of future light cones) uniquely determines the location of all points in L that contribute non-trivially to $f(L)$. Intuitively, this relates to the fact that for $d = 1$ the joint future of any two points has the same geometry as the light cone of an individual point (the unique earliest point in this joint future), but the geometries of these regions will differ for $d > 1$ as there is no longer a unique earliest point in the joint future of any two points.

3. Correspondence between causal inference and space-time geometry: We study the properties of affects relations and causal models, which can be embedded in a space-time compatibly, i.e., without violating no superluminal signalling (NSS). We show that an affects relation $X \vDash Y \mid \text{do}(Z)$ irreducible in the first and third arguments implies that all nodes in X and in Z are causes of some node in Y i.e., the set Y of nodes succeeds each node in X and Z relative to the *information-theoretic causal order*. However, we observe that when embedding these nodes in space-time, imposing that the affects relations respect NSS does not generally imply an analogous ordering of nodes relative to the *light-cone structure of the space-time*. We then prove a correspondence between (1) affects relations that are *not clustered in the third argument* but embedded compatibly in an arbitrary space-time and (2) arbitrary affects relations but embedded compatibly in a *conical space-time*. Specifically, we show that in both these cases, a clear ordering emerges between interventional arguments X and Z and the observational argument Y of an irreducible affects relation $X \vDash Y \mid \mathrm{do}(Z)$, with the former ordered before the latter relative to both the information-theoretic and spatio-temporal causal orders. As clustering implies fine-tuning, this suggests links between faithful causal models and conical space-times.

In summary, we found that the relativistic principle of NSS and purely informational principles of causal inference generally impose different ordering constraints on the relevant operational events (here, the arguments of an affects relation). However, our main result shows that restricting the space-time structure through conicality or restricting the causal models through faithfulness assumption both lead to the emergence of useful parallels between the relativistic and information-theoretic causal order relations. This provides further evidence towards the conjecture that the compatibly embeddable causal loops (which require fine-tuning) found in 1+1-Minkowski space-time [12] (which is not conical) may not be possible in higher spatial dimensions (where we have conicality), and also highlights that in conical space-times, one can often reduce statements regarding NSS for higher-order affects relations to equivalent statements in terms of the much simpler zeroth-order relations.

3 Outlook

Causal modelling and inference have widespread use in data driven disciplines, and order-theoretic properties of space-time are of interest in general relativity and quantum gravity approaches. Our work offers new insights and tools for investigating questions of relevance to these different communities, and may be of interest beyond quantum information and physics. These results shed light on foundational questions about the interface of information theory and space-time structure, and can inform the development of more robust causal inference techniques and algorithms capable of accounting for fine-tuning. Further, the relevance of fine-tuning in the security of cryptographic protocols (as highlighted in [10]) motivates potential applications of our results in more practical quantum scenarios.

In particular, the concept of clustering introduced here, captures a property that is necessary for the security of cryptographic protocols such as the one-time pad and secret sharing schemes [15, 16, 17], where information is distributed over multiple systems and cannot be recovered from a subset thereof. This is also similar to the desired properties of quantum error correcting codes [18, 19] that are intimately linked to quantum information-processing tasks in space-time such as summoning [20, 21, 22]. Integrating the theory-independent techniques of the affects framework [10] and our work, with insights from approaches that can describe the non-localisation of quantum information in space-time [23, 24, 25, 26] to develop causality-based characterisations of such quantum protocols remains a promising avenue for future work.

References

[1] P. Spirtes, C. Glymour, and R. Scheines. *Causation, Prediction, and Search*. Springer New York, 1993. ISBN: 9781461227489. DOI: 10.1007/978-1-4612-2748-9.

[2] J. Pearl. *Causality: Models, Reasoning and Inference*. 2nd ed. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511803161.

[3] D. R. Schoolmaster, C. R. Zirbel, and J. P. Cronin. "A graphical causal model for resolving species identity effects and biodiversity–ecosystem function correlations". In: *Ecology* 101.8 (June 2020). DOI: 10.1002/ecy.3070.

[4] Z. M. Laubach et al. "A biologist's guide to model selection and causal inference". In: *Proceedings of the Royal Society B: Biological Sciences* 288.1943 (Jan. 2021), p. 20202815. DOI: 10.1098/rspb.2020.2815.

[5] S. Kleinberg and G. Hripcsak. "A review of causal inference for biomedical informatics". In: *Journal of Biomedical Informatics* 44.6 (2011), pp. 1102–1112. ISSN: 1532-0464. DOI: 10.1016/j.jbi.2011.07.001.

[6] Y. Raita et al. "Big Data, Data Science, and Causal Inference: A Primer for Clinicians". In: *Frontiers in Medicine* 8 (July 2021). DOI: 10.3389/fmed.2021.678047.

[7] S. Arti, I. Hidayah, and S. S. Kusumawardhani. "Research Trend of Causal Machine Learning Method: A Literature Review". In: *IJID (International Journal on Informatics for Development)* 9.2 (Dec. 2020), pp. 111–118. DOI: 10.14421/ijid.2020.09208.

[8] J.-M. A. Allen et al. "Quantum Common Causes and Quantum Causal Models". In: *Phys. Rev. X* 7 (July 2017), p. 031021. DOI: 10.1103/PhysRevX.7.031021.

[9] J. Barrett, R. Lorenz, and O. Oreshkov. *Quantum Causal Models.* 2019. DOI: 10.48550/arXiv.1906.10726. arXiv: 1906.10726 [quant-ph].

[10] V. Vilasini and R. Colbeck. "General framework for cyclic and fine-tuned causal models and their compatibility with space-time". In: *Phys. Rev. A* 106 (3 Sept. 2022), p. 032204. DOI: 10.1103/PhysRevA.106.032204.

[11] V. Vilasini and R. Colbeck. *A causal modelling analysis of Bell scenarios in space-time: implications of jamming non-local correlations for relativistic causality principles.* 2023. DOI: 10.48550/arXiv.2311.18465. arXiv: 2311.18465 [quant-ph].

[12] V. Vilasini and R. Colbeck. "Impossibility of Superluminal Signaling in Minkowski Spacetime Does Not Rule Out Causal Loops". In: *Phys. Rev. Lett.* 129 (11 Sept. 2022), p. 110401. DOI: 10.1103/PhysRevLett.129.110401.

[13] V. Vilasini and R. Colbeck. *Information-processing in theories constrained by no superluminal causation vs no superluminal signalling.* 2024. DOI: 10.48550/arXiv.2402.12446. arXiv: 2402.12446 [quant-ph].

[14] F. Costa and S. Shrapnel. "Quantum causal modelling". In: *New Journal of Physics* 18.6 (June 2016), p. 063032. ISSN: 1367-2630. DOI: 10.1088/1367-2630/18/6/063032.

[15] M. Hillery, V. Bužek, and A. Berthiaume. "Quantum secret sharing". In: *Phys. Rev. A* 59 (3 Mar. 1999), pp. 1829–1834. DOI: 10.1103/PhysRevA.59.1829.

[16] D. Gottesman. "Theory of quantum secret sharing". In: *Phys. Rev. A* 61 (4 Mar. 2000), p. 042311. DOI: 10.1103/PhysRevA.61.042311.

[17] A. Karlsson, M. Koashi, and N. Imoto. "Quantum entanglement for secret sharing and secret splitting". In: *Phys. Rev. A* 59 (1 Jan. 1999), pp. 162–168. DOI: 10.1103/PhysRevA.59.162.

[18] D. E. Gottesman. "Stabiliser Codes and Quantum Error Correction". PhD thesis. California Institute of Technology, 1997. DOI: 10.48550/arXiv.2102.02393.

[19] R. Cleve, D. Gottesman, and H.-K. Lo. "How to Share a Quantum Secret". In: *Phys. Rev. Lett.* 83 (3 July 1999), pp. 648–651. DOI: 10.1103/PhysRevLett.83.648.

[20] A. Kent. "A no-summoning theorem in relativistic quantum theory". In: *Quantum Information Processing* 12.2 (July 2012), pp. 1023–1032. ISSN: 1573-1332. DOI: 10.1007/s11128-012-0431-6.

[21] A. Kent. "Quantum tasks in Minkowski space". In: *Classical and Quantum Gravity* 29.22 (Oct. 2012), p. 224013. ISSN: 1361-6382. DOI: 10.1088/0264-9381/29/22/224013.

[22] P. Hayden and A. May. "Summoning information in spacetime, or where and when can a qubit be?" In: *Journal of Physics A: Mathematical and Theoretical* 49.17 (Mar. 2016), p. 175304. ISSN: 1751-8121. DOI: 10.1088/1751-8113/49/17/175304.

[23] C. Portmann et al. "Causal Boxes: Quantum Information-Processing Systems Closed under Composition". In: *IEEE Transactions on Information Theory* (2017), pp. 1–1. ISSN: 1557-9654. DOI: 10.1109/tit.2017.2676805. URL: http://dx.doi.org/10.1109/TIT.2017.2676805.

[24] G. Chiribella and H. Kristjánsson. "Quantum Shannon theory with superpositions of trajectories". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 475.2225 (May 2019), p. 20180903. ISSN: 1471-2946. DOI: 10.1098/rspa.2018.0903. URL: http://dx.doi.org/10.1098/rspa.2018.0903.

[25] A. Vanrietvelde, H. Kristjánsson, and J. Barrett. "Routed quantum circuits". In: *Quantum* 5 (July 2021), p. 503. ISSN: 2521-327X. DOI: 10.22331/q-2021-07-13-503. URL: http://dx.doi.org/10.22331/q-2021-07-13-503.

[26] V. Vilasini and R. Renner. *Embedding cyclic causal structures in acyclic spacetimes: no-go results for process matrices.* Mar. 2022. DOI: 10.48550/arXiv.2203.11245. arXiv: 2203.11245 [quant-ph].