Bell Non-locality from Wigner Negativity in Qudit Stabilizer States

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As a key concept that distinguishes quantum from classical models, we study non-locality in qudit systems. Although non-locality has been extensively studied for qubits, some findings from their two-level counterparts do not generalize to higher-dimensional systems. In contrast to qubits, qudit stabilizer states cannot display Bell non-locality with Clifford operators [5, 7]. As a further testimony of their semi-classical nature, they have non-negative Gross's Wigner functions [2]. Since Wigner negativity has been shown to be equivalent to contextuality, a generalization of non-locality, in qudit systems [1], it is a necessary prerequisite for non-locality, either present in the state or in the measurement.

We propose a family of Bell inequalities on two qudits for any finite odd prime dimension d by constructing a bipartite Bell operator that consists of stabilizer elements of the qudit Bell state under the local adjoint action of a non-Clifford unitary operator. The Bell state maximally violates the corresponding Bell inequality as a result of the difference between the 1-norm and maximum norm and the Wigner negativity from the non-Clifford operators. The Bell inequality is a natural extension of the Clauser-Horne-Shimony-Holt (CHSH) inequality for qubits, which is a linear combination of Pauli operators under local rotation of the T-gate. Moreover, the Bell operator not only serves as a measure for the singlet fraction, but also quantifies the volume of Wigner negativity. We are able to adapt the Bell operator on multiple qudits such that given stabilizer state maximally violates it with similar implications as for the bipartite case. Additionally, we demonstrate deterministic violations and violations with a constant number of measurements for the bipartite Bell state, relying on operators innate to higher-dimensional systems than the qudit at hand.

The unitary qudit operators X and Z are a natural generalization of the qubit Pauli operators σ_x and σ_z. They fulfill $X|k\rangle = |(k+1) \bmod d\rangle$ and $Z|k\rangle = \omega^k|k\rangle$, with the d^{th} root of unity $\omega = \exp(2\pi \mathrm{i}/d)$, and the relation $Z^z X^x = \omega^{xz} X^x Z^z$ for $x, z \in \mathbb{Z}_d$. The general qudit Pauli operators are the Heisenberg-Weyl displacement operators $T_{(x,z)} = \omega^{2^{-1}xz} X^x Z^z$. By applying the Fourier transform, we obtain positive Hermitian operators

$$A_{(u_x,u_z)} = \frac{1}{d} \sum_{(v_x,v_z)\in\mathbb{Z}_d^{2n}} \omega^{u_z v_x - u_x v_z} T_{(v_x,v_z)}\,, \tag{1}$$

which form an orthonormal basis in the vector space of operators equipped with the Hilbert-Schmidt inner product $(A,B) := \mathrm{tr}\big(A^\dagger B\big)$. Gross' Wigner function is then defined by

$$W_{(u_x,u_z)}(\rho) := \frac{1}{d}\,\mathrm{tr}\left(A_{(u_x,u_z)}\rho\right)\,. \tag{2}$$

The Bell state $|\Phi\rangle = \sum_{k=0}^{d-1} |kk\rangle/\sqrt{d}$ is stabilized by the Pauli operators $T_{(x,z)} \otimes T_{(x,-z)}|\Phi\rangle = |\Phi\rangle$, and has a non-negative Wigner function. To induce Wigner negativity, we use an extension of the qubit T-gate to qudits introduced by Howard and Vala [6] who provide an analytic expression for diagonal non-Clifford unitary operators that map Pauli to Clifford operators, which we call *unitary cube operators*,

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$U_\nu = \sum_{k=0}^{d-1} \omega^{\nu_k} |k\rangle\langle k|$, with the third-order polynomial $\deg(\nu_k) = 3$ in the finite field $\mathbb{Z}_d$. The Bell state under the adjoint action of unitary cube operators has a Wigner function with negative values, $W^{\nu}_{v_1,v_2} := W_{v_1,v_2}\left(U_\nu \otimes \mathbb{1}\right)|\Phi\rangle\langle\Phi|\left(U_\nu^\dagger \otimes \mathbb{1}\right)$.

For $d > 3$, the Wigner function is determined by the character sum with a third-order odd polynomial. For general odd primes d, such polynomials are difficult to analyze analytically, but values a_1, a_3 such that $W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|) < 0$ always exist and can be found efficiently with an exhaustive brute-force search. Unitary operators beyond the unitary cube operators, for instance, from higher-degree polynomials over finite fields, can achieve larger Wigner negativity and enhance non-local violations. A measure of the amount of Wigner negativity of a state ρ is its volume $N[\rho] = (\sum_u |W_u(\rho)| - 1)/2$.

For the first Bell operator, we measure the operators that make up the Wigner function in (1) to exploit the Wigner negativity of the stabilizer states under unitary cube operators. To highlight the negative values, a favorable coefficient distinguishes the negative values from the positive ones, for which we use the Wigner function itself, resulting in the Bell operator

$$B_\nu = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} W^{\nu}_{v_1,v_2}\, U_\nu A_{v_1} U_\nu^\dagger \otimes A_{v_2}\,. \tag{3}$$

Since we measure a complete set of operators A_v, the lhv model assigns deterministic classical values $d\,\langle A_u\rangle_{\text{lhv}^*} = d^2 \delta_{u,a}$ for $a \in \mathbb{Z}_d^2$ to its local marginals (lhv*) and, hence, $\langle B_\nu\rangle_{\text{lhv}^*} = d^2\, W^{\nu}_{a_1,a_2}$. As a result, any lhv model can maximally achieve

$$\langle B_\nu\rangle_{\text{lhv}} \leq \max_{v_1,v_2 \in \mathbb{Z}_d^2} d^2 W^{\nu}_{v_1,v_2} =: B^{\max}_{\text{lhv}}\,, \quad \text{while} \quad \operatorname{tr}(B_\nu |\Phi\rangle\langle\Phi|) = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} d^2 (W^{\nu}_{v_1,v_2})^2 = 1\,,$$

is the expectation value for the Bell state. Since the eigenvalues of A_u are bounded by d^n (and here, $n = 2$), $B^{\max}_{\text{lhv}} < \operatorname{tr}(B_\nu |\Phi\rangle\langle\Phi|)$ from the Wigner negativity $N > 0$ and the norm inequality $\|\cdot\|_\infty \leq \|\cdot\|_1$ in compact spaces. The normalization of the Bell operator has been chosen such that $\operatorname{tr}(B_\nu \rho) = \langle\Phi|\rho|\Phi\rangle$, provides a measure for the singlet fraction with $\operatorname{tr}(B_\nu |\Phi\rangle\langle\Phi|) = 1$. Furthermore, it provides a lower bound for the volume of Wigner negativity $\operatorname{tr}(B_\nu \rho) \leq \operatorname{tr}(B_\nu \sigma) \leq B^{\max}_{\text{lhv}}(1 + 2N\left[(U_\nu \otimes \mathbb{1})\sigma(U_\nu^\dagger \otimes \mathbb{1})\right])$,

For a more compact Bell operator, we focus solely on the stabilizer elements, $T_{(x,z)} \otimes T_{(x,-z)}|\Phi\rangle = |\Phi\rangle$, which reduces the number of measurements from $(d+1)^2$ to $(d+1)$, and leads to

$$B'_\nu = \sum_{x,z,t \in \mathbb{Z}_d} W^{(\nu)}_{(x,z),(x,-z)}\, U_\nu A_{(x,z)} U_\nu^\dagger \otimes A_{(x,t-z)}\,. \tag{4}$$

The expectation values have the same form as for the full Bell operator, but the summation and maximum take only coefficients $x, z, t \in \mathbb{Z}_d$. The operator is also a measure of the singlet fraction and the volume of Wigner negativity.

For qutrits, all unitary operators defined by character polynomials are Clifford operators. Therefore, the third root of the characters, $(-1)^{1/9} = \omega^{1/3}$, is necessary in [6] but we can derive a Wigner function that yields equivalent results when applied to the Bell operators in Eqs. (3)-(6). However, operators with a spectrum beyond the qudit characters, in contrast to those whose eigenvalues are ω^a for some integer a, appear to achieve stronger Bell violations as showcased in [8] for qutrit GHZ states.

Likewise, we present a Bell operator with diagonal unitary operators $V_q = \sum_{k \in \mathbb{Z}_d} \omega^{kq}|k\rangle\langle k|$, where q is a non-integer rational number and $X_q = V_q X (V_q)^\dagger$. These operators have an obvious advantage, which is resorting to phases ω^q that are beyond the description of any local value assignment, which can only resort to characters ω^k. Then, the Bell operator

$$B_{(1/2)} = \frac{1}{d} \sum_{k \in \mathbb{Z}_d} X^k_{(1/2)} \otimes \left(X^k_{(1/2)} + \omega^{-k} X^k_{(-1/2)} \right) + X^k_{(-1/2)} \otimes \left(X^k_{(-1/2)} + \omega^{-k} X^k_{(1/2)} \right) , \tag{5}$$

leads to a Bell inequality $\langle B_{(1/2)} \rangle_{\text{lhv}} \leq 3$. In contrast, the Bell state has an expectation value $\langle \Phi | B | \Phi \rangle = 4$ that achieves a deterministic violation. One can even reduce the number of operators and only consider $k = 1, d-1$ in Eq. (5). Then, an lhv model can achieve $\langle B'_{(1/2)} \rangle_{\text{lhv}} \leq 3 + \cos(4\pi/d) < 4$. As a trade-off, the separation between the classical and quantum models grows smaller with increasing d.

Lastly, we generalize the Bell operator in Eq. (4) to any n-qudit stabilizer state $|S\rangle$ with elements $S_{\boldsymbol{u}} = \omega^{[\boldsymbol{a},\boldsymbol{u}]} T_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \Sigma \subset \mathbb{Z}_d^{2n}$. For a unitary cube operator U_ν acting on the first qudit,

$$B_S = \sum_{\boldsymbol{u} \in \Sigma, q \in \mathbb{Z}_d} W^\nu_{\boldsymbol{u}} \left(U_\nu A_{u_1 + (0,t)} U^\dagger_\nu \right) \bigotimes_{i=2}^{n} A_{u_i} . \tag{6}$$

The operator B_S is a measure for the overlap with the given stabilizer state under the condition that $((0, \mathbb{Z}_d)_1 \otimes (0,0)^{\otimes n-1}) \subset \Sigma$, which is exactly the case if $|S\rangle$ is entangled over the cut of the first qudit. Then, $\langle S | B_S | S \rangle = 1$, while $\langle B_S \rangle_{\text{lhv}} \leq d^n \max_{\boldsymbol{u} \in \Sigma} W^\nu_{\boldsymbol{u}} < 1$. A family of Bell operators, where the unitary cube operator U_ν acts on a different qudit, can detect if all qudits are entangled with any other qudit but does not expose genuine multipartite entanglement.

References

[1] Nicolas Delfosse, Cihan Okay, Juan Bermejo-Vega, Dan E. Browne & Robert Raussendorf (2017): *Equivalence between contextuality and negativity of the Wigner function for qudits*. New Journal of Physics 19, doi:10.1088/1367-2630/aa8fe3.

[2] D. Gross (2006): *Hudson's theorem for finite-dimensional quantum systems*. Journal of Mathematical Physics 47, doi:10.1063/1.2393152.

[3] Otfried Gühne & Géza Tóth (2009): *Entanglement detection*. Physics Reports 474(1), pp. 1–75, doi:https://doi.org/10.1016/j.physrep.2009.02.004.

[4] Mark Howard (2015): *Maximum nonlocality and minimum uncertainty using magic states*. Phys. Rev. A 91, p. 042103, doi:10.1103/PhysRevA.91.042103.

[5] Mark Howard, Eoin Brennan & Jiri Vala (2013): *Quantum contextuality with stabilizer states*. Entropy 15, pp. 2340–2362, doi:10.3390/e15062340.

[6] Mark Howard & Jiri Vala (2012): *Qudit versions of the qubit π/8 gate*. Physical Review A - Atomic, Molecular, and Optical Physics 86, doi:10.1103/PhysRevA.86.022316.

[7] Mark Howard, Joel Wallman, Victor Veitch & Joseph Emerson (2014): *Contextuality supplies the 'magic' for quantum computation*. Nature 510, doi:10.1038/nature13460.

[8] Jay Lawrence (2017): *Mermin inequalities for perfect correlations in many-qutrit systems*. Phys. Rev. A 95, p. 042123, doi:10.1103/PhysRevA.95.042123.