COMBINING CONTEXTUALITY AND CAUSALITY: A GAME SEMANTICS APPROACH

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Abstract. We develop an approach to combining contextuality with causality, which is general enough to cover causal background structure, adaptive measurement-based quantum computation, and causal networks. The key idea is to view contextuality as arising from a game played between Experimenter and Nature, allowing for causal dependencies in the actions of both the Experimenter (choice of measurements) and Nature (choice of outcomes).

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Introduction

Contextuality is a key non-classical feature of quantum theory. Besides its importance in quantum foundations, it has been linked to quantum advantage in information-processing tasks. It also arises beyond quantum mechanics, cf. [5].

We generalise contextuality to accommodate *causality* and *adaptivity*. These features may arise from:

- fundamental aspects of the physical setting, in particular the causal structure of spacetime;
- the causal structure of an experiment, where measurements are performed in some causal order, and moreover, which measurements are performed may depend on the outcomes of previous measurements;
- feed forward in measurement-based quantum computation (MBQC) [4], and more generally, adaptive computation.

Our objectives include:

- A more fine-grained analysis of contextuality. Signalling should be allowed from the causal past, i.e. the backward light cone, and thus no-signalling/no-disturbance should be imposed only from *outside* it. This in turn modifies the scope of classicality (non-contextuality), which now becomes relative to this weaker form of no-signalling constraints.
- A better connection with computational models such as circuits and MBQC. Explicitly representing causal flows of information, e.g. outputs of gates feeding into inputs of other gates, enables a deeper analysis of the relationships between contextuality and quantum advantage.

It turns out that capturing these different manifestations of causality and their interactions with contextuality is rather subtle. The perspective we adopt here is to view contextuality as a two-person game played between Experimenter and Nature. The Experimenter's moves are the measurements; i.e. the actions of the Experimenter are to choose the next measurement to be performed. Nature's moves are the outcomes. We can capture the various forms of causal dependency which may arise in terms of *strategies* for Experimenter or for Nature.

The game format is already familiar in the form of non-local games. There, the Verifier plays the role of the Experimenter, and Nature responds with outcomes according to the probability distributions corresponding to Alice–Bob strategies. Non-local games are one-shot games, with a single round of interaction. By considering more general games, causal structure can be incorporated.

Our treatment builds upon the sheaf-theoretic approach to contextuality [2]. A pleasing feature is that once one modifies the basic sheaf of events to take causal structure into account, the further definitions and treatment of contextuality follow *automatically*. This illustrates the advantages of a compositional and functorial approach.

As motivating examples, we take two basic kinds of scenarios. First, we consider variants of Bell scenarios with a causal background, i.e. a partial order on the sites or parties in the scenario, as studied by Gogioso and Pinzani [6, 7] in their causal refinement of the sheaf-theoretic approach. Secondly, we consider the Anders–Browne construction [3] where the experimenter exploits the contextuality of GHZ state *adaptively*, i.e. using the possibility of making measurement choices depend on their causal past, to promote a parity classical computer to implement arbitrary Boolean functions by MBQC. Section 3 in the full paper contains a detailed presentation of these two examples. In them, causality plays a very different role in relation to contextuality: in one, it is imposed by Nature in the form of a causal background against which the contextual behaviour plays out; in the other, it is imposed by the Experimenter to achieve computational effects (adaptive computation).

This paper extends the scope of previous work in several directions (see Section 2 of the paper for more details) First, we allow more general dependencies of events on their prior causal histories. In particular, the choice of which measurement to perform can depend on previous outcomes as well as on which measurements have been performed. This is an important feature of MBQC ("feedforward"), and more generally of adaptive computation. Secondly, we extend general contextuality scenarios with causality, not just the non-locality Bell scenarios as in the Gogioso–Pinzani (GP) approach. Finally, and most subtly, we recognise the different roles played by Nature and Experimenter in their causal interactions, highlighting an important difference between causal background and adaptivity.

An interesting feature of our approach, in common with that of Gogioso–Pinzani, is that it proceeds essentially by modifying the sheaf of events from [2] to reflect the refined signalling constraints in the presence of causality. Once this has been done, the remainder of the analysis of contextuality follows exactly the same script as in [2]. In particular, the appropriate definition of empirical model, the relaxed no-signalling constraints, and the notion of classicality/non-contextuality follow automatically.

Game semantics of causality

We conceptualise the dual nature of causality as a two-person game, played between Experimenter and Nature:

- Experimenter's moves are measurements to be performed;
- Nature's moves are the outcomes.

By formalising this, we develop a theory of causal contextuality that recovers, as special cases: the usual theory of contextuality in the "flat" case, the Gogioso–Pinzani theory of non-locality in a causal background, MBQC with adaptive computation, classical causal networks, and more.

Measurement scenarios. We review some basic ingredients of the sheaf-theoretic formulation of contextuality (see e.g. [2]). A *(flat) measurement scenario* is a pair (X, O), where: X is a set of *measurements*, and $O = \{O_x\}_{x \in X}$ is the set of possible *outcomes* for each measurement.

An *event* has the form (x, o), where $x \in X$ and $o \in O_x$. It corresponds to the measurement x being performed, with outcome o being observed. Given a set of events s, its domain is the set of measurements performed:

$$\mathrm{dom}(s) := \pi_1 s = \{x \mid \exists o. (x, o) \in s\}.$$

We say that s is *consistent* if $(x, y), (x, y') \in s$ implies $y = y'$. In this case, s defines a function from the measurements in its domain to outcomes. A consistent set of events is a *section*. We define the *event sheaf* $\mathcal{E}$ over sets of measurements: for each set $U \subseteq X$ of measurements, $\mathcal{E}(U)$ is the set of sections whose domain is U; when $U \subseteq V$, there is a restriction map $\mathcal{E}(V) \to \mathcal{E}(U)$. Functoriality of these restriction maps formalises the no-disturbance condition, or "generalised no-signalling", at the level of deterministic models. No-disturbance for general probabilistic (or possibilistic) models then follows automatically when we compose with the appropriate distribution monad, cf. [2].

The sheaf property of the event sheaf – that compatible families of local sections glue together to yield unique global sections – corresponds to the fact that *deterministic models are non-contextual.* When we pass to distributions over the event sheaf, the sheaf property no longer holds, and this is exactly how contextuality arises. More precisely, we extend the measurement scenario to a *contextuality scenario* by specifying a cover of X; a failure of the sheaf property with respect to this cover constitutes a witness to contextuality.

Causal measurement scenarios. Our general strategy to accommodate causality is to modify the definition of the event sheaf. After this, we essentially follow the same script as above to give an account of contextuality in the causal setting. A similar procedure is followed in [6, 7].

A *causal measurement scenario* is a tuple $M = (X, O, \vdash)$, where the additional ingredient is an *enabling relation* that expresses causal constraints. The intended interpretation of $s \vdash x$, where $s \in \bigcup_{U \subseteq X} \mathcal{E}(U)$ is a consistent set of events and $x \in X$ a measurement, is that it is possible to perform x after the events in s have occurred. Note that this constraint refers to the measurement outcomes as well as the measurements that have been performed. This allows adaptive behaviours to be described.

Given such a causal measurement scenario M, we use it to generate a set of *histories.* A history is a set of events that can happen in a causally consistent fashion. We associate each measurement x with a unique event occurrence, so histories are required to be consistent. To formalise this, first define the *accessibility relation* $\rhd$ between consistent sets of events s and measurements x: $s \rhd x$ if and only if $x \notin \mathrm{dom}(s)$ and for some $t \subseteq s$, $t \vdash x$. The intuition is that x may be performed if the events in s have occurred. Now, $\mathcal{H}(M)$, the set of histories over M, is defined inductively as the least family H of consistent sets of events which contains the empty set and is closed under accessibility, meaning that if $s \in H$ and $s \rhd x$, then for all $o \in O_x$, $s \cup \{(x, o)\} \in H$. Note that if a measurement can be performed, then any of its outcomes may occur, forming a valid history.

Nature strategies. We regard a causal measurement scenario as specifying a game between Experimenter and Nature. Events (x, o) correspond to the Experimenter choosing a measurement x, and Nature responding with outcome o. The histories correspond to the *plays* or runs of the game. We consider the notion of strategy, focusing first on the player Nature, whose strategies one may think of as hidden variables. This is in line with the usual discussion of contextuality, where the experimenter may freely choose which measurements to perform. Later on, we shall also consider the parallel notion of strategy for Experimenter, which can express adaptivity.

A *strategy for Nature* over the game M as a set of histories $\sigma \subseteq \mathcal{H}(M)$ satisfying the following conditions:

- σ is downwards closed: if $s, t \in \mathcal{H}(M)$ and $s \subseteq t \in \sigma$, then $s \in \sigma$.
- σ is deterministic and total: if $s \in \sigma$ and $s \rhd x$, then there is a unique $o \in O_x$ such that $s \cup \{(x, o)\} \in \sigma$.

Thus at any position s reachable under the strategy σ, the strategy determines a unique response to any measurement that can be chosen by the Experimenter.

The presheaf of strategies. Given a causal measurement scenario $M = (X, O, \vdash)$ and a set of measurements $U \subseteq X$, we define M_U, the restriction of M to U, as the causal measurement scenario $(U, \{O_x\}_{x \in U}, \vdash_U)$, where $s \vdash_U x$ iff $s \vdash x$ and $\mathrm{dom}(s) \cup \{x\} \subseteq U$. Note that $M_X = M$.

This lets us define a presheaf $\Gamma : \mathcal{P}(X)^{\mathsf{op}} \to \mathbf{Set}$ of strategies associated with the scenario M: for each $U \subseteq X$, $\Gamma(U)$ is the set of strategies for M_U; given $U \subseteq V$, the restriction map is given by $\sigma \mapsto \sigma|_U$.

Causal contextuality

A *causal contexuality scenario* is a structure $(M, \mathcal{C})$, where $M = (X, O, \vdash)$ is a causal measurement scenario and $\mathcal{C}$ is a cover of X, i.e. a family $\mathcal{C} = \{C_i\}_{i \in I}$ of subsets of measurements $C_i \subseteq X$ satisfying $\bigcup \mathcal{C} = \bigcup_{i \in I} C_i = X$. We work with the presheaf Γ of strategies over M, as described in the previous section.

An *empirical model* on the scenario $(M, \mathcal{C})$ is a compatible family for the presheaf $\mathcal{D}_R\Gamma$ over the cover $\mathcal{C} = \{C_i\}_{i \in I}$, where $\mathcal{D}_R$ is the distribution monad for a semiring R (e.g. the non-negative reals for discrete probability distributions). That is, it is a family $\{e_i\}_{i \in I}$, where $e_i \in \mathcal{D}_R\Gamma(C_i)$, subject to the compatibility conditions: for all $i, j \in I$, $e_i|_{C_i \cap C_j} = e_j|_{C_i \cap C_j}$. Each distribution e_i assigns probabilities to the strategies over M_{C_i}, i.e. to those strategies over M that only perform measurements drawn from the context C_i. As usual, the compatibility conditions require that the marginal distributions agree. This follows the definition of empirical model in [2], replacing the event sheaf by the presheaf of strategies.

The empirical model is *causally non-contextual* if this compatible family extends to a global section of the presheaf $\mathcal{D}_R\Gamma$, i.e. if there is a distribution $d \in \mathcal{D}_R\Gamma(X)$ such that, for all $i \in I$, $d|_{C_i} = e_i$.

If a causal contextuality scenario is finite, then so is the set of histories and therefore that of strategies. The causally non-contextual models thus form a convex polytope, the convex hull of the empirical models on $(M, \mathcal{C})$ corresponding to deterministic strategies $\sigma \in \Gamma(X)$. This is in keeping with the usual setup of "flat" non-locality and contextuality (i.e. without causality), where such classical polytopes are studied. The classicality of a given model, i.e. membership in this polytope, can be checked by linear programming; and this also suggests a generalisation of the contextual fraction [1] to the causal setting.

Further results

We briefly summarise the further results. Details can be found in the indicated sections of the full paper.

- We show that the usual "flat" contextuality scenarios and the Gogioso–Pinzani causal Bell scenarios are both special cases of our approach (Section 6).
- We study the sheaf property for the presheaf property. We verify that it holds for covers consisting of sets of measurements that are causally secure in an appropriate sense (Section 7).[1]
- We introduce a notion of Experimenter strategy, and show that this allows us to capture adaptive computation, exemplified by the Anders–Browne construction (Section 8).

We also discuss further directions for this work (Section 9). One of these, on temporal correlations, has already been substantially developed, and is the subject of another submission to QPL.

References

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[1] Since full details are not provided in the paper, we elaborate slightly here. The idea is to define a relation on measurements from the enabling, and require this to be well-founded. This then defines an strict order on measurements, and the good subsets are those which are downwards closed in this order.